Derived Differential Geometry

Lecture 1 of 14: Background material in algebraic geometry and category theory

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Summer 2015

These slides, and references, etc., available at
`http://people.maths.ox.ac.uk/~joyce/DDG2015`

Plan of talk:

1. Different kinds of spaces in algebraic geometry

Algebraic geometry studies spaces built using algebras of functions. Here are the main classes of spaces studied by algebraic geometers, in order of complexity, and difficulty of definition:

- Smooth varieties (e.g. Riemann surfaces, or algebraic complex manifolds such as $\mathbb{CP}^n$. Smooth means nonsingular.)
- Varieties (at their most basic, algebraic subsets of $\mathbb{C}^n$ or $\mathbb{CP}^n$. Can have singularities, e.g. $xy = 0$ in $\mathbb{C}^2$, singular at $(0,0)$.)
- Schemes (can be non-reduced, e.g. the scheme $x^2 = 0$ in $\mathbb{C}$ is not the same as the scheme $x = 0$ in $\mathbb{C}$.)
- Algebraic spaces (étale locally modelled on schemes.)
- Stacks. Each point $x \in X$ has a stabilizer group $\mathrm{Iso}(x)$, finite for Deligne–Mumford stacks, algebraic group for Artin stacks.
- Higher stacks.
- Derived stacks, including derived schemes.

1.1. The definition of schemes

Fix a field $\mathbb{K}$, e.g. $\mathbb{K} = \mathbb{C}$. An *affine $\mathbb{K}$-scheme* $X = \operatorname{Spec} A$ is basically a commutative $\mathbb{K}$-algebra A, but regarded as a geometric space in the following way. As a set, define X to be the set of all prime ideals $I \subset A$. If J is an ideal of A, define $V(J) \subseteq X$ to be the set of prime ideals $I \subseteq A$ with $J \subseteq I$. Then $\mathcal{T} = \{V(J) : J$ is an ideal in $A\}$ is a topology on X, the *Zariski topology*.

We can regard each $f \in A$ as a 'function' on X, where $f(I) = f + I$ in the quotient algebra A/I. For the subset $X(\mathbb{K}) \subseteq X$ of $\mathbb{K}$-*points* I with $A/I \cong \mathbb{K}$, f gives a genuine function $X(\mathbb{K}) \to \mathbb{K}$.

Thus, we have a topological space X called the *spectrum* $\operatorname{Spec} A$ of A, equipped with a sheaf of $\mathbb{K}$-algebras $\mathcal{O}_X$, and A is the algebra of functions on X.

A general $\mathbb{K}$-scheme X is a topological space X with a sheaf of $\mathbb{K}$-algebras $\mathcal{O}_X$, such that X may be covered by open sets $U \subseteq X$ with $(U, \mathcal{O}_X|_U)$ isomorphic to $\operatorname{Spec} A$ for some $\mathbb{K}$-algebra A.

Example 1.1

$\mathbb{C}^n$ is an affine $\mathbb{C}$-scheme, the spectrum of the polynomial algebra $A = \mathbb{C}[x_1, \ldots, x_n]$. Given polynomials $p_1, \ldots, p_k \in \mathbb{C}[x_1, \ldots, x_n]$, we can define an affine $\mathbb{C}$-subscheme $X \subseteq \mathbb{C}^n$ as the zero locus of $p_1, \ldots, p_k$, the spectrum of $B = \mathbb{C}[x_1, \ldots, x_n]/(p_1, \ldots, p_k)$. The $\mathbb{C}$-points $X(\mathbb{C})$ are $(x_1, \ldots, x_n) \in \mathbb{C}^n$ with $p_1(x_1, \ldots, x_n) = \cdots = p_k(x_1, \ldots, x_n) = 0$. Note that the (nonreduced) scheme $x^2 = 0$ in $\mathbb{C}$ is not the same as the scheme $x = 0$ in $\mathbb{C}$, as the algebras $\mathbb{C}[x]/(x^2) = \mathbb{C}\langle 1, x\rangle$ and $\mathbb{C}[x]/(x) = \mathbb{C}\langle 1\rangle$ are different.

To take a similar approach to manifolds M in differential geometry, we should consider the $\mathbb{R}$-algebra $C^\infty(M)$ of smooth functions $f : M \to \mathbb{R}$, and reconstruct M as the set of ideals $I \subset C^\infty(M)$ with $C^\infty(M)/I \cong \mathbb{R}$. In lecture 3 we will see that $C^\infty(M)$ is not just an $\mathbb{R}$-algebra, it has an algebraic structure called a *C^∞-ring*, and M is a scheme over C^∞-rings, a *C^∞-scheme*.

1.2. Some basics of category theory

To prepare for moduli functors and stacks, we first explain some ideas from category theory.

Definition

A *category* $\mathcal{C}$ consists of the following data:

- A family $\mathrm{Obj}(\mathcal{C})$ of *objects* $X, Y, Z, \ldots$ of $\mathcal{C}$. (Actually $\mathrm{Obj}(\mathcal{C})$ is a *class*, like a set but possibly larger.)
- For all objects X, Y in $\mathcal{C}$, a set $\mathrm{Hom}(X, Y)$ of *morphisms* f, written $f : X \to Y$.
- For all objects X, Y, Z in $\mathcal{C}$, a *composition map* $\circ : \mathrm{Hom}(Y, Z) \circ \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z)$, written $g \circ f : X \to Z$ for morphisms $f : X \to Y$ and $g : Y \to Z$. It is associative, $(h \circ g) \circ f = h \circ (g \circ f)$.
- For all objects X in $\mathcal{C}$ an *identity morphism* $\mathrm{id}_X \in \mathrm{Hom}(X, X)$, with $f \circ \mathrm{id}_X = \mathrm{id}_Y \circ f = f$ for all $f : X \to Y$.

Categories are everywhere in mathematics – whenever you have a class of mathematical objects, and a class of maps between them, you generally get a category. For example:

- The category **Sets** with objects sets, and morphisms maps.
- The category **Top** with objects topological spaces $X, Y, \ldots$ and morphisms continuous maps $f : X \to Y$.
- The category **Man** of smooth manifolds and smooth maps.
- The category $\mathbf{Sch}_{\mathbb{K}}$ of schemes over a field $\mathbb{K}$.

Definition

A category $\mathcal{C}$ is a *subcategory* of a category $\mathscr{D}$, written $\mathcal{C} \subset \mathscr{D}$, if $\mathrm{Obj}(\mathcal{C}) \subseteq \mathrm{Obj}(\mathscr{D})$, and for all $X, Y \in \mathrm{Obj}(\mathcal{C})$ we have $\mathrm{Hom}_{\mathcal{C}}(X, Y) \subseteq \mathrm{Hom}_{\mathscr{D}}(X, Y)$, and composition and identities in $\mathcal{C}, \mathscr{D}$ agree on $\mathrm{Hom}_{\mathcal{C}}(-,-)$. It is a *full* subcategory if $\mathrm{Hom}_{\mathcal{C}}(X, Y) = \mathrm{Hom}_{\mathscr{D}}(X, Y)$ for all X, Y in $\mathcal{C}$.

The category $\mathbf{Sch}^{\mathbf{aff}}_{\mathbb{K}}$ of affine $\mathbb{K}$-schemes is a full subcategory of $\mathbf{Sch}_{\mathbb{K}}$.

Functors between categories

Functors are the natural maps between categories.

Definition

Let $\mathcal{C}, \mathscr{D}$ be categories. A *functor* $F : \mathcal{C} \to \mathscr{D}$ consists of the data:

- A map $F : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathscr{D})$.
- Maps $F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$ for all $X, Y \in \mathrm{Obj}(\mathcal{C})$, with $F(g \circ f) = F(g) \circ F(f)$ for all composable f, g in $\mathcal{C}$ and $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all X in $\mathcal{C}$.

- The *identity functor* $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ maps $X \mapsto X$ and $f \mapsto f$.
- There is a ‘forgetful functor’ F : **Man** $\to$ **Top** taking a manifold X to its underlying topological space $F(X)$, and a smooth map $f : X \to Y$ to its underlying continuous map.
- If $\mathcal{C} \subset \mathscr{D}$ is a subcategory, the *inclusion functor* $i : \mathcal{C} \hookrightarrow \mathscr{D}$.
- For $k \geqslant 0$, H_k : **Top** $\to$ **AbGp** (abelian groups) maps a topological space X to its k^{th} homology group $H_k(X; \mathbb{Z})$.

Our definition of functors are sometimes called *covariant functors*, in contrast to *contravariant functors* $F : \mathcal{C} \to \mathscr{D}$ which reverse the order of composition, $F(g \circ f) = F(f) \circ F(g)$, such as the cohomology functors $H^k : \textbf{Top} \to \textbf{AbGp}$. We prefer to write contravariant functors as (covariant) functors $F : \mathcal{C}^{\mathrm{op}} \to \mathscr{D}$, where $\mathcal{C}^{\mathrm{op}}$ is the *opposite category* to $\mathcal{C}$, the same as $\mathcal{C}$ but with order of composition reversed. For example, in scheme theory the *spectrum functor* maps $\mathrm{Spec} : (\textbf{Alg}_{\mathbb{K}})^{\mathrm{op}} \to \textbf{Sch}^{\textbf{aff}}_{\mathbb{K}} \subset \textbf{Sch}_{\mathbb{K}}$, where $\textbf{Alg}_{\mathbb{K}}$ is the category of $\mathbb{K}$-algebras.

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What is derived geometry?
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Algebraic spaces and (higher) stacks

Natural transformations and natural isomorphisms

There is also a notion of morphism between functors:

Definition

Let $\mathcal{C}, \mathscr{D}$ be categories, and $F, G : \mathcal{C} \to \mathscr{D}$ be functors. A *natural transformation* η from F to G, written $\eta : F \Rightarrow G$, assigns the data of a morphism $\eta(X) : F(X) \to G(X)$ in $\mathscr{D}$ for all objects X in $\mathcal{C}$, such that $\eta(Y) \circ F(f) = G(f) \circ \eta(X) : F(X) \to G(Y)$ for all morphisms $f : X \to Y$ in $\mathcal{C}$.
We call η a *natural isomorphism* if $\eta(X)$ is an isomorphism (invertible morphism) in $\mathscr{D}$ for all X in $\mathcal{C}$.
Given natural transformations $\eta : F \Rightarrow G$, $\zeta : G \Rightarrow H$, the *composition* $\zeta \odot \eta : F \Rightarrow H$ is $(\zeta \odot \eta)(X) = \zeta(X) \circ \eta(X)$ for X in $\mathcal{C}$.
The *identity transformation* $\mathrm{id}_F : F \Rightarrow F$ is $\mathrm{id}_F(X) = \mathrm{id}_{F(X)} : F(X) \to F(X)$ for all X in $\mathcal{C}$.

Note that in the 'category of categories' $\mathfrak{Cat}$, we have objects categories $\mathcal{C}, \mathscr{D}, \ldots$, and morphisms (or 1-morphisms), functors $F, G : \mathcal{C} \to \mathscr{D}$, but also 'morphisms between morphisms' (or 2-morphisms), natural transformations $\eta : F \Rightarrow G$. This is our first example of a 2-*category*, defined in lecture 4.

In category theory, it is often important to think about when things are 'the same'. For objects X, Y in a category $\mathcal{C}$, there are two notions of when X, Y are 'the same': equality $X = Y$, and isomorphism $X \cong Y$, i.e. there are morphisms $f : X \to Y$, $g : Y \to X$ with $g \circ f = \mathrm{id}_X$, $f \circ g = \mathrm{id}_Y$. Usually isomorphism is better.

For functors $F, G : \mathcal{C} \to \mathscr{D}$, there are two notions of when F, G are 'the same': equality $F = G$, and natural isomorphism $F \cong G$, that is, there exists a natural isomorphism $\eta : F \Rightarrow G$. Usually natural isomorphism, the weaker, is better.

Equivalence of categories

For categories $\mathcal{C}, \mathscr{D}$, there are three notions of when $\mathcal{C}, \mathscr{D}$ are 'the same': strict equality $\mathcal{C} = \mathscr{D}$; strict isomorphism $\mathcal{C} \cong \mathscr{D}$, that is, there exist functors $F : \mathcal{C} \to \mathscr{D}$, $G : \mathscr{D} \to \mathcal{C}$ with $G \circ F = \mathrm{id}_{\mathcal{C}}$, $F \circ G = \mathrm{id}_{\mathscr{D}}$; and equivalence:

Definition

An *equivalence* between categories $\mathcal{C}, \mathscr{D}$ consists of functors $F : \mathcal{C} \to \mathscr{D}$, $G : \mathscr{D} \to \mathcal{C}$ and natural isomorphisms $\eta : G \circ F \Rightarrow \mathrm{id}_{\mathcal{C}}$, $\zeta : F \circ G \Rightarrow \mathrm{id}_{\mathscr{D}}$. We say that G is a *quasi-inverse* for F, and write $\mathcal{C} \simeq \mathscr{D}$ to mean that $\mathcal{C}, \mathscr{D}$ are equivalent.

Usually equivalence of categories, the weakest, is the best notion of when categories $\mathcal{C}, \mathscr{D}$ are 'the same'.

The Yoneda embedding

Let $\mathcal{C}, \mathscr{D}$ be categories. Then $\mathrm{Fun}(\mathcal{C}, \mathscr{D})$ is a category with objects functors $F : \mathcal{C} \to \mathscr{D}$, morphisms natural transformations $\eta : F \Rightarrow G$, composition $\zeta \odot \eta$, and identities id_F. A natural transformation $\eta : F \Rightarrow G$ is a natural isomorphism if and only if it is an isomorphism in $\mathrm{Fun}(\mathcal{C}, \mathscr{D})$.

Definition

Let $\mathcal{C}$ be any category. Then $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets})$ is also a category. Define a functor $Y_{\mathcal{C}} : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets})$ called the *Yoneda embedding* by, for each X in $\mathcal{C}$, taking $Y_{\mathcal{C}}(X)$ to be the functor $\mathrm{Hom}(-, X) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Sets}$ mapping $Y \mapsto \mathrm{Hom}(Y, X)$ on objects $Y \in \mathcal{C}$, and mapping $\circ f : \mathrm{Hom}(Z, X) \to \mathrm{Hom}(Y, X)$ for all morphisms $f : Y \to Z$ in $\mathcal{C}$; and for each morphism $e : W \to X$ in $\mathcal{C}$, taking $Y_{\mathcal{C}}(e) : Y_{\mathcal{C}}(W) \to Y_{\mathcal{C}}(X)$ to be the natural transformation $e \circ : \mathrm{Hom}(-, W) \to \mathrm{Hom}(-, X)$.

The Yoneda Lemma

The *Yoneda Lemma* says $Y_{\mathcal{C}} : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets})$ is a *full* and *faithful* functor, i.e. the maps $Y_{\mathcal{C}} : \mathrm{Hom}_{\mathcal{C}}(W, X) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets})}(Y_{\mathcal{C}}(W), Y_{\mathcal{C}}(X))$ are injective and surjective.
Call a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Sets}$ *representable* if F is naturally isomorphic to $Y_{\mathcal{C}}(X) = \mathrm{Hom}(-, X)$ for some $X \in \mathcal{C}$, which is then unique up to isomorphism. Write $\mathrm{Rep}(\mathcal{C})$ for the full subcategory of representable functors in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets})$. Then $Y_{\mathcal{C}} : \mathcal{C} \to \mathrm{Rep}(\mathcal{C})$ is an equivalence of categories.
Basically, the idea here is that we should understand objects X in $\mathcal{C}$, up to isomorphism, by knowing the sets $\mathrm{Hom}(Y, X)$ for all $Y \in \mathcal{C}$, and the maps $\circ f : \mathrm{Hom}(Z, X) \to \mathrm{Hom}(Y, X)$ for all morphisms $f : Y \to Z$ in $\mathcal{C}$.
If $\mathcal{C} \subset \mathscr{D}$ is a subcategory, there is a functor $\mathscr{D} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets})$ mapping $X \mapsto \mathrm{Hom}(\mathcal{C}, X)$ for $X \in \mathscr{D}$.

(Affine) schemes as functors $\mathbf{Alg}_{\mathbb{K}} \to \mathbf{Sets}$

Since $\mathrm{Spec} : (\mathbf{Alg}_{\mathbb{K}})^{\mathrm{op}} \to \mathbf{Sch}^{\mathbf{aff}}_{\mathbb{K}}$ is an equivalence of categories, and $(\mathbf{Alg}_{\mathbb{K}})^{\mathrm{op}}$ is equivalent to the full subcategory of representable functors in $\mathrm{Fun}(\mathbf{Alg}_{\mathbb{K}}, \mathbf{Sets})$, we see that $\mathbf{Sch}^{\mathbf{aff}}_{\mathbb{K}}$ is equivalent to the full subcategory of representable functors in $\mathrm{Fun}(\mathbf{Alg}_{\mathbb{K}}, \mathbf{Sets})$.
There is also a natural functor $\mathbf{Sch}_{\mathbb{K}} \to \mathrm{Fun}(\mathbf{Alg}_{\mathbb{K}}, \mathbf{Sets})$, mapping a scheme X to the functor $A \mapsto \mathrm{Hom}_{\mathbf{Sch}_{\mathbb{K}}}(\mathrm{Spec}\, A, X)$. This functor is full and faithful because, as X can be covered by open subschemes $\mathrm{Spec}\, A \subseteq X$, we can recover X up to isomorphism from the collection of morphisms $\mathrm{Spec}\, A \to X$ for $A \in \mathbf{Alg}_{\mathbb{K}}$.
Thus, $\mathbf{Sch}_{\mathbb{K}}$ is equivalent to a full subcategory of $\mathrm{Fun}(\mathbf{Alg}_{\mathbb{K}}, \mathbf{Sets})$.
Since we consider equivalent categories to be 'the same', *we can identify* $\mathbf{Sch}_{\mathbb{K}}$ *with this subcategory of* $\mathrm{Fun}(\mathbf{Alg}_{\mathbb{K}}, \mathbf{Sets})$, and *we can consider schemes to be special functors* $\mathbf{Alg}_{\mathbb{K}} \to \mathbf{Sets}$.

1.3. Moduli spaces and moduli functors

Moduli spaces are a hugely important subject in both algebraic and differential geometry. They were also the motivation for inventing most of the classes of spaces we are discussing – algebraic spaces, stacks, derived stacks, ... – as interesting moduli spaces had these structures, and simpler spaces were not adequate to describe them.
Suppose we want to study some class of geometric objects X up to isomorphism, e.g. Riemann surfaces of genus g. Write $\mathcal{M}$ for the set of isomorphism classes $[X]$ of such X. A set on its own is boring, so we would like to endow $\mathcal{M}$ with some geometric structure which captures properties of families $\{X_t : t \in T\}$ of the objects X we are interested in. For example, if we have a notion of continuous deformation $X_t : t \in [0,1]$ of objects X, then we should give $\mathcal{M}$ a topology such that the map $[0,1] \to \mathcal{M}$ mapping $t \mapsto [X_t]$ is continuous for all such families $X_t : t \in [0,1]$.

We would like the geometric structure we put on $\mathcal{M}$ to be as strong as possible (e.g. for Riemann surfaces not just a topological space, but a complex manifold, or a $\mathbb{C}$-scheme) to capture as much information as we can about families of objects $\{X_t : t \in T\}$.
To play the moduli space game, we must ask three questions:

(A) What kind of geometric structure should we try to put on $\mathcal{M}$ (e.g. topological space, complex manifold, $\mathbb{K}$-scheme, ...)?
(B) Does $\mathcal{M}$ actually have this structure?
(C) If it does, can we describe $\mathcal{M}$ in this class of geometric spaces completely, or approximately (e.g. if $\mathcal{M}$ is a complex manifold, can we compute its dimension, and Betti numbers $b^k(M)$)?

There are two main reasons people study moduli spaces. The first is *classification*: when you study some class of geometric objects X (e.g. vector bundles on curves), people usually consider that if you can fully describe the moduli space $\mathcal{M}$ (with whatever geometric structure is appropriate), then you have classified such objects.

The second reason is *invariants*. There are many important areas of mathematics (e.g. Gromov–Witten invariants) in which to study some space S (e.g. a symplectic manifold) we form moduli spaces $\mathcal{M}$ of secondary geometric objects X associated to S (e.g. J-holomorphic curves in S), and then we define invariants $I(S)$ by 'counting' $\mathcal{M}$, to get a number, a homology class, etc.
We want the invariants $I(S)$ to have nice properties (e.g. to be independent of the choice of almost complex structure J on S).
For this to hold it is *essential* that the geometric structure on $\mathcal{M}$ be of a very special kind (e.g. a compact oriented manifold), and the 'counting' be done in a very special way.
Theories of this type include Donaldson, Donaldson–Thomas, Gromov–Witten, and Seiberg–Witten invariants, Floer homology theories, and Fukaya categories in symplectic geometry.
This is actually a major motivation for Derived Differential Geometry: compact, oriented derived manifolds or derived orbifolds can be 'counted' in this way to define invariants.

Moduli schemes and representable functors

In algebraic geometry there is a standard method for defining moduli spaces as schemes, due to Grothendieck. Suppose we want to form a moduli scheme $\mathcal{M}$ of some class of geometric objects X over a field $\mathbb{K}$. Suppose too that we have a good notion of family $\{X_t : t \in T\}$ of such objects X over a base $\mathbb{K}$-scheme T. We then define a *moduli functor* $F : \mathbf{Alg}_{\mathbb{K}} \to \mathbf{Sets}$, by for each $A \in \mathbf{Alg}_{\mathbb{K}}$

$$F(A)=\bigl\{\text{iso. classes } [X_t : t\in\operatorname{Spec} A] \text{ of families } \{X_t : t\in\operatorname{Spec} A\}\bigr\},$$

and for each morphism $f : A \to A'$ in $\mathbf{Alg}_{\mathbb{K}}$, we define $F(f) : F(A) \to F(A')$ by

$$F(f) : [X_t : t \in \operatorname{Spec} A] \longmapsto [X_{\operatorname{Spec}(f)t'} : t' \in \operatorname{Spec}(A')].$$

If there exists a $\mathbb{K}$-scheme $\mathcal{M}$ (always unique up to isomorphism) such that F is naturally isomorphic to $\operatorname{Hom}(\operatorname{Spec} -, \mathcal{M})$, we say F is a *representable functor*, and $\mathcal{M}$ is a (*fine*) *moduli scheme*.

1.4. Algebraic spaces and (higher) stacks
When schemes are not enough

Unfortunately there are lots of interesting moduli problems in which one can define a moduli functor $F : \mathbf{Alg}_{\mathbb{K}} \to \mathbf{Sets}$, but F is not representable, and no moduli scheme exists.
Sometimes one can find a $\mathbb{K}$-scheme $\mathcal{M}$ which is a 'best approximation' to F (a *coarse moduli scheme*). But often, to describe the moduli space $\mathcal{M}$, we have to move out of schemes, into a larger class of spaces.
The simplest such enlargement is *algebraic spaces*, which are defined to be functors $F : \mathbf{Alg}_{\mathbb{K}} \to \mathbf{Sets}$ which can be presented as the quotient $\mathcal{M}/\sim$ of a scheme $\mathcal{M}$ by an étale equivalence relation $\sim$. For example, moduli spaces of simple complexes $\mathcal{E}^\bullet$ of coherent sheaves on a smooth projective $\mathbb{K}$-scheme S are algebraic spaces.

Introduction to stacks

A moduli $\mathbb{K}$-scheme $\mathcal{M}$ or moduli functor $F : \mathbf{Alg}_{\mathbb{K}} \to \mathbf{Sets}$ classifies objects X up to isomorphism, so that $\mathbb{K}$-points of $\mathcal{M}$ are isomorphism classes $[X]$ of objects X. For each X we have a group $\mathrm{Iso}(X)$ of isomorphisms $i : X \to X$.

Usually, if $\mathrm{Iso}(X)$ is nontrivial, then F is not representable, and $\mathcal{M}$ does not exist as either a scheme or an algebraic space. Roughly, the reason is that we should expect $\mathcal{M}$ to be modelled near $[X]$ on a quotient $[\mathcal{N}/\mathrm{Iso}(X)]$ for a scheme $\mathcal{N}$, but schemes and algebraic spaces are not closed under quotients by groups (though see GIT).

Stacks are a class of geometric spaces $\mathcal{M}$ in which the geometric structure at each point $[X] \in \mathcal{M}$ remembers the group $\mathrm{Iso}(X)$. They include *Deligne–Mumford stacks*, in which the groups $\mathrm{Iso}(X)$ are finite, and *Artin stacks*, in which the $\mathrm{Iso}(X)$ are algebraic $\mathbb{K}$-groups. For almost all classical moduli problems a moduli stack exists, even when a moduli scheme does not.

Groupoids and stacks

A *groupoid* is a category $\mathcal{C}$ in which all morphisms are isomorphisms. They form a category **Groupoids** in which objects are groupoids, and morphisms are functors between them.

Any set S can be regarded as a groupoid with objects $s \in S$, and only identity morphisms. This gives a full and faithful functor $\mathbf{Sets} \to \mathbf{Groupoids}$, so $\mathbf{Sets} \subset \mathbf{Groupoids}$ is a full subcategory.

You can also map (small) groupoids to sets by sending $\mathcal{C}$ to the set S of isomorphism classes in $\mathcal{C}$.

A *stack* is defined to be a functor $F : \mathbf{Alg}_{\mathbb{K}} \to \mathbf{Groupoids}$ satisfying some complicated conditions. Since (affine) schemes and algebraic spaces can all be regarded as functors $F : \mathbf{Alg}_{\mathbb{K}} \to \mathbf{Sets}$, and $\mathbf{Sets} \subset \mathbf{Groupoids}$, we can consider (affine) schemes and algebraic spaces as functors $F : \mathbf{Alg}_{\mathbb{K}} \to \mathbf{Groupoids}$, and then they are special examples of stacks.

Stacks as moduli functors

As for moduli schemes, there is a standard method for defining moduli stacks. Suppose we want to form a moduli stack $\mathcal{M}$ of some class of geometric objects X over a field $\mathbb{K}$. We define a *moduli functor* $F : \mathbf{Alg}_{\mathbb{K}} \to \mathbf{Groupoids}$, by for each $A \in \mathbf{Alg}_{\mathbb{K}}$

$$F(A) = \bigl\{\text{groupoid of families } \{X_t : t \in \operatorname{Spec} A\}, \text{ with morphisms isomorphisms of such families}\bigr\},$$

and for each morphism $f : A \to A'$ in $\mathbf{Alg}_{\mathbb{K}}$, we define $F(f) : F(A) \to F(A')$ to be the functor of groupoids mapping

$$F(f) : \{X_t : t \in \operatorname{Spec} A\} \longmapsto \{X_{\operatorname{Spec}(f)t'} : t' \in \operatorname{Spec}(A')\}.$$

If F satisfies the necessary conditions, then F *is* the moduli stack.

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With some practice you can treat stacks as geometric spaces – they have points, a topology, ‘atlases’ which are schemes, and so on. Stacks X are often locally modelled on quotients Y/G, for Y a scheme, and G a group which is finite for Deligne–Mumford stacks, and an algebraic group for Artin stacks.

Above we saw that categories form a 2-category $\mathfrak{Cat}$, with objects categories, 1-morphisms functors, and 2-morphisms natural transformations. As groupoids are special categories, **Groupoids** is also a 2-category. Since all natural transformations of groupoids are natural isomorphisms, all 2-morphisms in **Groupoids** are invertible, i.e. it is a (2,1)-*category*.

Stacks $\subset \operatorname{Fun}(\mathbf{Alg}_{\mathbb{K}}, \mathbf{Groupoids})$ also form a (2,1)-category, with 2-morphisms defined using natural isomorphisms of groupoids.

Higher stacks

There are some moduli problems for which even stacks are not general enough. A typical example would be moduli spaces $\mathcal{M}$ of complexes $\mathcal{E}^\bullet$ in the derived category $D^b \operatorname{coh}(S)$ of coherent sheaves on a smooth projective scheme S. The point is that $\mathcal{M}$ classifies complexes $\mathcal{E}^\bullet$ not up to isomorphism, but up to a weaker notion of quasi-isomorphism. Really $D^b \operatorname{coh}(S)$ is an ∞-category. For such moduli problems we need *higher stacks*, which are functors $F : \mathbf{Alg}_{\mathbb{K}} \to \mathbf{SSets}$. Here $\mathbf{SSets}$ is the (∞-)category of *simplicial sets*, which are generalizations of groupoids, so that $\mathbf{Sets} \subset \mathbf{Groupoids} \subset \mathbf{SSets}$. Higher stacks form an ∞-*category*, meaning that there are not just objects, 1-morphisms, and 2-morphisms, but n-morphisms for all $n = 1, 2, \ldots$.

Derived Differential Geometry

Lecture 2 of 14: What is derived geometry?

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These slides, and references, etc., available at
`http://people.maths.ox.ac.uk/~joyce/DDG2015`

Plan of talk:

2. What is derived geometry?

Derived geometry is the study of 'derived' spaces. It has two versions: Derived Algebraic Geometry (DAG), the study of derived schemes and derived stacks, and Derived Differential Geometry (DDG), the study of derived smooth real (i.e. C^∞) spaces, including derived manifolds and derived orbifolds.
DAG is older and more developed. It has a reputation for difficulty and abstraction, with foundational documents running to 1000's of pages (Lurie, Toën–Vezzosi). DDG is a new subject, just beginning, with few people working in it so far.
Today we begin with an introduction to DAG, to give some idea of what 'derived' spaces are, why they were introduced, and what they are useful for. An essential point is that derived geometry happens in *higher categories* (e.g. 2-categories or ∞-categories).

2.1. Derived schemes and derived stacks

In §1 we saw that in classical algebraic geometry, we have spaces

affine schemes ⊂ schemes ⊂ algebraic spaces ⊂ stacks ⊂ higher stacks,

which can be defined as classes of functors $F : \mathbf{Alg}_{\mathbb{K}} \to \mathbf{Sets}$ or **Groupoids** or **SSets**, where **Sets** ⊂ **Groupoids** ⊂ **SSets**.
Such a space X is completely described by knowing the family (set, or groupoid, or simplicial set) of all morphisms $f : \operatorname{Spec} A \to X$, for all $\mathbb{K}$-algebras A, plus the family of all commutative triangles

$$\begin{tikzcd} \operatorname{Spec} A' \arrow[d, "\operatorname{Spec}\alpha"] \arrow[dr, "f'"] & \\ \operatorname{Spec} A \arrow[r, "f"] & X, \end{tikzcd}$$

for all morphisms of $\mathbb{K}$-algebras $\alpha : A \to A'$.

To do derived geometry, instead of enlarging the target category **Sets**, we enlarge the domain category $\mathbf{Alg}_{\mathbb{K}}$. So, a *derived stack* over a field $\mathbb{K}$ is defined to be a functor $\mathbf{F} : \mathbf{CDGAlg}_{\mathbb{K}} \to \mathbf{SSets}$ satisfying complicated conditions, where $\mathbf{CDGAlg}_{\mathbb{K}}$ is the category of *commutative differential graded* $\mathbb{K}$*-algebras* (*cdgas*) in degrees $\leqslant 0$, which we explain shortly. An alternative definition, essentially equivalent when $\operatorname{char}\mathbb{K} = 0$, uses functors $\mathbf{F} : \mathbf{SAlg}_{\mathbb{K}} \to \mathbf{SSets}$, where $\mathbf{SAlg}_{\mathbb{K}}$ is the category of *simplicial* $\mathbb{K}$*-algebras*.
One might guess that derived schemes should be functors $\mathbf{F} : \mathbf{CDGAlg}_{\mathbb{K}} \to \mathbf{Sets}$, and derived stacks functors $\mathbf{F} : \mathbf{CDGAlg}_{\mathbb{K}} \to \mathbf{Groupoids}$, and derived higher stacks functors $\mathbf{F} : \mathbf{CDGAlg}_{\mathbb{K}} \to \mathbf{SSets}$. In fact only functors $\mathbf{F} : \mathbf{CDGAlg}_{\mathbb{K}} \to \mathbf{SSets}$ are considered. This is because Derived Algebraic Geometers always make things maximally complicated.

Any $\mathbb{K}$-algebra A can be regarded as a cdga $A^\bullet$ concentrated in degree 0, giving a full subcategory $\mathbf{Alg}_{\mathbb{K}} \subset \mathbf{CDGAlg}_{\mathbb{K}}$. Thus, any functor $\mathbf{F} : \mathbf{CDGAlg}_{\mathbb{K}} \to \mathbf{SSets}$ restricts to a functor $t_0(\mathbf{F}) = \mathbf{F}|_{\mathbf{Alg}_{\mathbb{K}}} : \mathbf{Alg}_{\mathbb{K}} \to \mathbf{SSets}$, called the *classical truncation* of $\mathbf{F}$. If $\mathbf{F}$ is a derived scheme, or derived Deligne–Mumford / Artin stack, or derived stack, then $t_0(\mathbf{F})$ is a scheme, or Deligne–Mumford / Artin stack, or higher stack, respectively.
So, derived stacks do not allow us to study a larger class of moduli problems, as algebraic spaces/stacks/higher stacks do. Instead, they give us a *richer geometric structure* on the moduli spaces we already knew about in classical algebraic geometry.
This is because a derived stack $\mathbf{X}$ knows about all morphisms $\operatorname{Spec} A^\bullet \to \mathbf{X}$ for all cdgas $A^\bullet$, but the corresponding classical stack $X = t_0(\mathbf{X})$ only knows about all morphisms $\operatorname{Spec} A \to X$ for all $\mathbb{K}$-algebras A, which is less information.

2.2. Commutative differential graded $\mathbb{K}$-algebras

Cdgas in derived geometry replace algebras in classical geometry.

Definition

Let $\mathbb{K}$ be a field. A *commutative differential graded $\mathbb{K}$-algebra* (*cdga*) $A^\bullet = (A^*, \mathrm{d})$ in degrees $\leqslant 0$ consists of a $\mathbb{K}$-vector space $A^* = \bigoplus_{k=0}^{-\infty} A^k$ graded in degrees $0, -1, -2, \ldots,$ together with $\mathbb{K}$-bilinear multiplication maps $\cdot : A^k \times A^l \to A^{k+l}$ for all $k, l \leqslant 0$ which are associative and supercommutative (i.e. $\alpha \cdot \beta = (-1)^{kl} \beta \cdot \alpha$ for all $\alpha \in A^k$, $\beta \in A^l$), an identity $1 \in A^0$ with $1 \cdot \alpha = \alpha \cdot 1 = \alpha$ for all $\alpha \in A^*$, and $\mathbb{K}$-linear differentials $\mathrm{d} : A^k \to A^{k+1}$ for all $k < 0$, which satisfy $\mathrm{d}^2 = 0$ and the Leibnitz rule $\mathrm{d}(\alpha \cdot \beta) = (\mathrm{d}\alpha) \cdot \beta + (-1)^k \alpha \cdot (\mathrm{d}\beta)$ for all $\alpha \in A^k$ and $\beta \in A^l$.

Write $H^k(A^\bullet) = \operatorname{Ker}(\mathrm{d} : A^k \to A^{k+1}) / \operatorname{Im}(\mathrm{d} : A^{k-1} \to A^k)$ for the cohomology of $A^\bullet$. Then $H^*(A^\bullet)$ is a graded $\mathbb{K}$-algebra, and $H^0(A^\bullet)$ an ordinary $\mathbb{K}$-algebra.

Example 2.1 (Our main example of cdgas and derived schemes)

Let $m, n \geqslant 0$, and consider the free graded $\mathbb{C}$-algebra $A^* = \mathbb{C}[x_1, \ldots, x_m; y_1, \ldots, y_n]$ generated by commutative variables $x_1, \ldots, x_m$ in degree 0, and anti-commutative variables $y_1, \ldots, y_n$ in degree -1. Then $A^k = \mathbb{C}[x_1, \ldots, x_m] \otimes_{\mathbb{C}} (\Lambda^{-k}\mathbb{C}^n)$ for $k = 0, -1, \ldots, -n$, and $A^k = 0$ otherwise.
Let $p_1, \ldots, p_n \in \mathbb{C}[x_1, \ldots, x_m]$ be complex polynomials in $x_1, \ldots, x_m$. Then as A^* is free, there are unique maps $\mathrm{d} : A^k \to A^{k+1}$ satisfying the Leibniz rule, such that $\mathrm{d}y_i = p_i(x_1, \ldots, x_m)$ for $i = 1, \ldots, n$. Also $\mathrm{d}^2 = 0$, so $A^\bullet = (A^*, \mathrm{d})$ is a cdga. We have $H^0(A^\bullet) = \mathbb{C}[x_1, \ldots, x_m]/(p_1, \ldots, p_n)$, where $(p_1, \ldots, p_n)$ is the ideal generated by $p_1, \ldots, p_n$. Hence $\mathrm{Spec}\, H^0(A^\bullet)$ is the subscheme of $\mathbb{C}^m$ defined by $p_1 = \cdots = p_n = 0$.
We interpret the derived scheme $\mathbf{Spec}\, A^\bullet$ as the derived subscheme of $\mathbb{C}^m$ defined by the equations $p_1 = \cdots = p_n = 0$.

What data does a derived scheme remember?

Consider the solutions X of $p_1 = \cdots = p_n = 0$ in $\mathbb{C}^m$ as (a) a variety, (b) a scheme, and (c) a derived scheme.
The variety X remembers only the set of solutions $(x_1, \ldots, x_m)$ in $\mathbb{C}^m$. So, for example, $x = 0$ and $x^2 = 0$ are the same variety in $\mathbb{C}$.
The scheme X remembers the ideal $(p_1, \ldots, p_n)$, so $x = 0$, $x^2 = 0$ are different schemes in $\mathbb{C}$ as $(x), (x^2)$ are distinct ideals in $\mathbb{C}[x]$.
But schemes forget dependencies between $p_1, \ldots, p_n$. So, for example, $x^2 = y^2 = 0$ with $n = 2$ and $x^2 = y^2 = x^2 + y^2 = 0$ with $n = 3$ are the same scheme in $\mathbb{C}^2$.
The derived scheme $\mathbf{X}$ *remembers information about the dependencies between* $p_1, \ldots, p_n$. For example $x^2 = y^2 = 0$ and $x^2 = y^2 = x^2 + y^2 = 0$ are different derived schemes in $\mathbb{C}^2$, as the two cdgas $A^\bullet, \tilde{A}^\bullet$ have $H^{-1}(A^\bullet) \not\cong H^{-1}(\tilde{A}^\bullet)$. In this case, $\mathbf{X}$ has a well-defined *virtual dimension* $\mathrm{vdim}\, \mathbf{X} = m - n$.

Bézout's Theorem and derived Bézout's Theorem

Let C, D be projective curves in $\mathbb{CP}^2$ of degrees m, n. If the classical scheme $X = C \cap D$ has dimension 0, then Bézout's Theorem says that the number of points in X counted with multiplicity (i.e. length(X)) is mn. But if $\dim X \neq 0$, counterexamples show you cannot recover mn from X.
Now consider the derived intersection $\mathbf{X} = C \cap D$. It is a proper, quasi-smooth derived scheme with $\mathrm{vdim}\,\mathbf{X} = 0$, even if $\dim X = 1$, and so has a 'virtual count' $[\mathbf{X}]_{\mathrm{virt}} \in \mathbb{Z}$, which is mn.
This is a derived version of Bézout's Theorem, without the transversality hypothesis $\dim C \cap D = 0$. It is possible as $\mathbf{X}$ remembers more about how C, D intersect. This illustrates:

General Principles of Derived Geometry

- Transversality is often not needed in derived geometry.
- Derived geometry is useful for Bézout-type 'counting' problems.

Patching together local models

Let X be a (say separated) classical $\mathbb{K}$-scheme. Then we can cover X by Zariski open subschemes $\mathrm{Spec}\,A \cong U \subseteq X$. Given two such $\mathrm{Spec}\,A \cong U$, $\mathrm{Spec}\,\tilde{A} \cong \tilde{U}$, we can compare them easily on the overlap $U \cap \tilde{U}$: there exist $f \in A$, $\tilde{f} \in \tilde{A}$ such that $U \cap \tilde{U}$ is identified with $\{f \neq 0\} \subseteq \mathrm{Spec}\,A$ and $\{\tilde{f} \neq 0\} \subseteq \mathrm{Spec}\,\tilde{A}$, and there is a canonical isomorphism of $\mathbb{K}$-algebras $A[f^{-1}] \cong \tilde{A}[\tilde{f}^{-1}]$, where $A[f^{-1}] = A[x]/(xf-1)$ is the $\mathbb{K}$-algebra obtained by inverting f in A.
For a derived scheme $\mathbf{X}$, really $\mathbf{X}$ is a functor $\mathbf{CDGAlg}_{\mathbb{K}} \to \mathbf{SSets}$, but we can at least pretend that $\mathbf{X}$ is a space covered by Zariski open $\mathbf{Spec}\,A^\bullet \cong \mathbf{U} \subseteq \mathbf{X}$. Given two $\mathbf{Spec}\,A^\bullet \cong \mathbf{U}$, $\mathbf{Spec}\,\tilde{A}^\bullet \cong \tilde{\mathbf{U}}$, we can find $f \in A^0$, $\tilde{f} \in \tilde{A}^0$ such that $\mathbf{U} \cap \tilde{\mathbf{U}}$ is identified with $\{f \neq 0\} \subseteq \mathbf{Spec}\,A^\bullet$ and $\{\tilde{f} \neq 0\} \subseteq \mathbf{Spec}\,\tilde{A}^\bullet$. However, in general we do *not* have $A^\bullet[f^{-1}] \cong \tilde{A}^\bullet[\tilde{f}^{-1}]$ in $\mathbf{CDGAlg}_{\mathbb{K}}$. Instead, $A^\bullet[f^{-1}], \tilde{A}^\bullet[\tilde{f}^{-1}]$ are only *equivalent* cdgas, in a weak sense.

The problem is that $\mathbf{CDGAlg}_{\mathbb{K}}$ is really the *wrong category*. A *quasi-isomorphism* is a morphism $f : A^\bullet \to \tilde A^\bullet$ in $\mathbf{CDGAlg}_{\mathbb{K}}$ such that $H^*(f) : H^*(A^\bullet) \to H^*(\tilde A^\bullet)$ is an isomorphism on cohomology. The correct statement is that $A^\bullet[f^{-1}], \tilde A^\bullet[\tilde f^{-1}]$ should be isomorphic in a 'localized' category $\mathbf{CDGAlg}_{\mathbb{K}}[\mathcal{Q}^{-1}]$ in which all quasi-isomorphisms in $\mathbf{CDGAlg}_{\mathbb{K}}$ have inverses. This is difficult to work with, and should really be an ∞-category.

General Principles of Derived Geometry

- You can usually give nice local models for 'derived' spaces $\mathbf{X}$. However, the local models are glued together on overlaps not by isomorphisms, but by some mysterious equivalence relation.
- We often study categories $\mathcal{C}$ of differential graded objects $A^\bullet$, in which quasi-isomorphisms $\mathcal{Q}$ are to be inverted. The resulting $\mathcal{C}[\mathcal{Q}^{-1}]$ must be treated as an ∞-category, as too much information is lost by the ordinary category.

2.3. Fibre products

To explain why we need higher categories in derived geometry, we discuss fibre products in (ordinary) categories.

Definition

Let $\mathcal{C}$ be a category, and $g : X \to Z$, $h : Y \to Z$ be morphisms in $\mathcal{C}$. A *fibre product* (W, e, f) for g, h in $\mathcal{C}$ consists of an object W and morphisms $e : W \to X$, $f : W \to Y$ in $\mathcal{C}$ with $g \circ e = h \circ f$, with the *universal property* that if $e' : W' \to X$, $f' : W' \to Y$ are morphisms in $\mathcal{C}$ with $g \circ e' = h \circ f'$, then there is a unique morphism $b : W' \to W$ with $e' = e \circ b$ and $f' = f \circ b$.
We write $W = X \times_{g,Z,h} Y$ or $W = X \times_Z Y$.
In general, fibre products may or may not exist. If a fibre product exists, it is unique up to canonical isomorphism.

Given a fibre product $W = X \times_{g,Z,h} Y$, the commutative diagram

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "e"] & Y \arrow[d, "h"] \\ X \arrow[r, "g"] & Z \end{tikzcd}$$

is called a *Cartesian square*. Some examples:

- All fibre products exist in $\mathbf{Sch}_{\mathbb{K}}$.
- All fibre products $W = X \times_{g,Z,h} Y$ exist in **Top**. We can take $W = \{(x, y) \in X \times Y : g(x) = h(y)\}$, with the subspace topology.
- Not all fibre products exist in **Man**. If $g : X \to Z$, $h : Y \to Z$ are *transverse* then a fibre product $W = X \times_{g,Z,h} Y$ exists in **Man** with $\dim W = \dim X + \dim Y - \dim Z$.

Intersections of subschemes, or submanifolds, are examples of fibre products. If $C, D \subseteq S$ are $\mathbb{K}$-subschemes of a $\mathbb{K}$-scheme S, then by the $\mathbb{K}$-subscheme $C \cap D$, we actually mean the fibre product $C \times_{i,S,j} D$ in $\mathbf{Sch}_{\mathbb{K}}$, with $i : C \hookrightarrow S$, $j : D \hookrightarrow S$ the inclusions.
Recall our 'derived Bézout's Theorem'. We claimed that given curves $C, D \subset \mathbb{CP}^2$ of degree m, n, there is a 'derived intersection' $\mathbf{X} = C \cap D$, which is quasi-smooth with dimension $\mathrm{vdim}\,\mathbf{X} = 0$, and has a 'virtual count' $[\mathbf{X}]_{\mathrm{virt}} \in \mathbb{Z}$, which is mn.
This statement *cannot be true* if $\mathbf{X}$ is the fibre product $C \times_{i,\mathbb{CP}^2,j} D$ in an ordinary category $\mathbf{dSch}_{\mathbb{C}}$ of derived $\mathbb{C}$-schemes. For example, if $C = D$ (or if C is a component of D), then in an ordinary category we must have $C \times_{\mathbb{CP}^2} D = C$, so that $\mathrm{vdim}\,\mathbf{X} = 1$.
However, it can be true if $\mathbf{dSch}_{\mathbb{C}}$ is a *higher category* (e.g. an ∞-category, or a 2-category), and fibre products in $\mathbf{dSch}_{\mathbb{C}}$ satisfy a more complicated universal property involving higher morphisms.

General Principles of Derived Geometry

- Derived geometric spaces should form higher categories (e.g. ∞-categories, or 2-categories), not ordinary categories.

In fact any higher category $\mathcal{C}$ has a *homotopy category* $\mathrm{Ho}(\mathcal{C})$, which is an ordinary category, where objects $\mathbf{X}$ of $\mathrm{Ho}(\mathcal{C})$ are objects of $\mathcal{C}$, and morphisms $[\mathbf{f}] : \mathbf{X} \to \mathbf{Y}$ in $\mathrm{Ho}(\mathcal{C})$ are 2-isomorphism classes of 1-morphisms $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ in $\mathcal{C}$. So we can reduce to ordinary categories, but this loses too much information.

- The 'correct' fibre products (etc.) in $\mathcal{C}$ satisfy universal properties in $\mathcal{C}$ involving higher morphisms. This does not work in $\mathrm{Ho}(\mathcal{C})$, where no universal property holds.
- In $\mathrm{Ho}(\mathcal{C})$, morphisms $[\mathbf{f}] : \mathbf{X} \to \mathbf{Y}$ are *not local in* $\mathbf{X}$. That is, if $\mathbf{U}, \mathbf{V} \subseteq \mathbf{X}$ are open with $\mathbf{X} = \mathbf{U} \cup \mathbf{V}$, then $[\mathbf{f}]$ is not determined by $[\mathbf{f}]|_{\mathbf{U}}$ and $[\mathbf{f}]|_{\mathbf{V}}$. To determine $\mathbf{f}$ up to 2-isomorphism you need to know the *choice* of 2-isomorphism $(\mathbf{f}|_{\mathbf{U}})|_{\mathbf{U}\cap\mathbf{V}} \to (\mathbf{f}|_{\mathbf{V}})|_{\mathbf{U}\cap\mathbf{V}}$, not just the existence.

2.4. Outlook on Derived Differential Geometry

There are several versions of 'derived manifolds' and 'derived orbifolds' in the literature, in order of increasing simplicity:

- Spivak's ∞-category $\mathbf{DerMan_{Spi}}$ of derived manifolds (2008).
- Borisov–Noël's ∞-category $\mathbf{DerMan_{BN}}$ of derived manifolds (2011,2012), which is equivalent to $\mathbf{DerMan_{Spi}}$.
- My d-manifolds and d-orbifolds (2010–2012), which form strict 2-categories $\mathbf{dMan}, \mathbf{dOrb}$.
- My M-Kuranishi spaces and Kuranishi spaces (2014), which form a category $\mathbf{MKur}$ and a weak 2-category $\mathbf{Kur}$.

In fact the (M-)Kuranishi space approach is motivated by earlier work by Fukaya, Oh, Ohta and Ono in symplectic geometry (1999,2009-) whose 'Kuranishi spaces' are really a prototype kind of derived orbifold, from before the invention of DAG.

The first version, Spivak's $\mathbf{DerMan_{Spi}}$, was an application of Jacob Lurie's DAG machinery in differential geometry. It is complicated and difficult to work with. Borisov–Noël gave an equivalent (as an ∞-category) but simpler definition $\mathbf{DerMan_{BN}}$. D-manifolds $\mathbf{dMan}$ are nearly a 2-category truncation of $\mathbf{DerMan_{Spi}}$, $\mathbf{DerMan_{BN}}$; Borisov defines a 2-functor $\pi_2(\mathbf{DerMan_{BN}}) \to \mathbf{dMan}$ identifying equivalence classes of objects, and surjective on 2-isomorphism classes of 1-morphisms. There are equivalences of weak 2-categories $\mathbf{dMan} \simeq \mathbf{Kur_{trG}}$ and of (homotopy) categories $\mathrm{Ho}(\mathbf{dMan}) \simeq \mathbf{MKur} \simeq \mathrm{Ho}(\mathbf{Kur_{trG}})$, where $\mathbf{Kur_{trG}} \subset \mathbf{Kur}$ is the 2-category of M-Kuranishi spaces with trivial orbifold groups.
For practical purposes, the five models $\mathbf{DerMan_{Spi}}, \mathbf{DerMan_{BN}}, \mathbf{dMan}, \mathbf{MKur}, \mathbf{Kur_{trG}}$ of derived manifolds are all equivalent, e.g. equivalence classes of objects in all five are in natural bijection.
This course will mainly discuss the simplest models $\mathbf{dMan}, \mathbf{dOrb}, \mathbf{MKur}, \mathbf{Kur}$ of derived manifolds and derived orbifolds.

For $\mathbf{DerMan_{Spi}}, \mathbf{DerMan_{BN}}$, little has been done beyond the original definitions. For $\mathbf{dMan}, \mathbf{dOrb}$ (also for $\mathbf{MKur}, \mathbf{Kur}$) there is a well developed differential geometry, studying immersions, submersions, transverse fibre products, orientations, bordism, virtual cycles, definition from differential-geometric data, etc.
The 'derived geometry' in $\mathbf{dMan}, \mathbf{dOrb}, \mathbf{MKur}, \mathbf{Kur}$ is, by the standards of Derived Algebraic Geometry, very simple. The theory uses 2-categories, which are much simpler than any form of ∞-category, and uses ordinary sheaves rather than homotopy sheaves. This is possible because of nice features of the differential-geometric context: the existence of partitions of unity, and the Zariski topology being Hausdorff.
The theory is still long and complicated for other reasons: firstly, the need to do algebraic geometry over C^∞-rings, and secondly, to define categories of derived manifolds and derived orbifolds *with boundary*, and *with corners*, which are needed for applications.

Properties of d-manifolds

A d-manifold $\mathbf{X}$ is a topological space X with a geometric structure. A d-manifold $\mathbf{X}$ has a *virtual dimension* $\mathrm{vdim}\,\mathbf{X} \in \mathbb{Z}$, which can be negative. If $x \in X$ then there is a *tangent space* $T_x\mathbf{X}$ and an *obstruction space* $O_x\mathbf{X}$, both finite-dimensional over $\mathbb{R}$ with $\dim T_x\mathbf{X} - \dim O_x\mathbf{X} = \mathrm{vdim}\,\mathbf{X}$.

Manifolds **Man** embed in **dMan** as a full (2-)subcategory. A d-manifold $\mathbf{X}$ is (equivalent to) an ordinary manifold if and only if $O_x\mathbf{X} = 0$ for all $x \in X$.

A 1-morphism of d-manifolds $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ is a continuous map $f : X \to Y$ with extra structure. If $x \in X$ with $f(x) = y$, then $\mathbf{f}$ induces functorial linear maps $T_x\mathbf{f} : T_x\mathbf{X} \to T_x\mathbf{Y}$ and $O_x\mathbf{f} : O_x\mathbf{X} \to O_y\mathbf{Y}$.

Fibre products of d-manifolds

Recall that smooth maps of manifolds $g : X \to Z$, $h : Y \to Z$ are *transverse* if for all $x \in X$, $y \in Y$ with $g(x) = h(y) = z \in Z$, then $T_xg \oplus T_yh : T_xX \oplus T_yY \to T_zZ$ is surjective. If g, h are transverse then a fibre product $W = X \times_{g,Z,h} Y$ exists in **Man**, with $\dim W = \dim X + \dim Y - \dim Z$.

Similarly, 1-morphisms $\mathbf{g} : \mathbf{X} \to \mathbf{Z}$, $\mathbf{h} : \mathbf{Y} \to \mathbf{Z}$ in **dMan** are *d-transverse* if for all $x \in X$, $y \in Y$ with $g(x) = h(y) = z \in Z$, then $O_x\mathbf{g} \oplus O_y\mathbf{h} : O_x\mathbf{X} \oplus O_y\mathbf{Y} \to O_z\mathbf{Z}$ is surjective.

Theorem 2.2

Suppose $\mathbf{g} : \mathbf{X} \to \mathbf{Z}$, $\mathbf{h} : \mathbf{Y} \to \mathbf{Z}$ are d-transverse 1-morphisms in **dMan**. *Then a fibre product $\mathbf{W} = \mathbf{X} \times_{\mathbf{g},\mathbf{Z},\mathbf{h}} \mathbf{Y}$ exists in the 2-category* **dMan**, *with $\mathrm{vdim}\,\mathbf{W} = \mathrm{vdim}\,\mathbf{X} + \mathrm{vdim}\,\mathbf{Y} - \mathrm{vdim}\,\mathbf{Z}$.*

Note that the fibre product $\mathbf{W} = \mathbf{X} \times_{\mathbf{Z}} \mathbf{Y}$ is characterized by a universal property involving 2-morphisms in **dMan**, which has no analogue in the ordinary category $\mathrm{Ho}(\mathbf{dMan})$. So we need a 2-category (or other higher category) for Theorem 2.2 to work.
D-transversality is a weak assumption. For example, if $\mathbf{Z}$ is a manifold then $O_z\mathbf{Z} = 0$ for all z, and any $\mathbf{g}, \mathbf{h}$ are d-transverse, so:

Corollary 2.3

Suppose $\mathbf{g} : \mathbf{X} \to Z$, $\mathbf{h} : \mathbf{Y} \to Z$ *in are* 1*-morphisms in* **dMan**, *with* Z *a manifold. Then a fibre product* $\mathbf{W} = \mathbf{X} \times_{\mathbf{g},Z,\mathbf{h}} \mathbf{Y}$ *exists in* **dMan**, *with* $\mathrm{vdim}\,\mathbf{W} = \mathrm{vdim}\,\mathbf{X} + \mathrm{vdim}\,\mathbf{Y} - \dim Z$.

This is really useful. For instance, if $g : X \to Z$, $h : Y \to Z$ are smooth maps of manifolds then a fibre product $\mathbf{W} = X \times_{g,Z,h} Y$ exists in **dMan** *without any transversality assumptions at all*.

General Principles of Derived Geometry

- Transversality is often not needed in derived geometry.

Application to moduli spaces

Almost any moduli space used in any enumerative invariant problem over $\mathbb{R}$ or $\mathbb{C}$ has a d-manifold or d-orbifold structure, natural up to equivalence. There are truncation functors to d-manifolds and d-orbifolds from structures currently used — FOOO Kuranishi spaces, polyfolds, $\mathbb{C}$-schemes or Deligne–Mumford $\mathbb{C}$-stacks with obstruction theories.
Combining these truncation functors with known results gives d-manifold/d-orbifold structures on many moduli spaces.
If $P(u) = 0$ is a nonlinear elliptic equation on a compact manifold, then the moduli space $\mathcal{M}$ of solutions u has the structure of a d-manifold $\boldsymbol{\mathcal{M}}$, where if $u \in \mathcal{M}$ is a solution and $\mathcal{L}_u P : C^{k+d,\alpha}(E) \to C^{k,\alpha}(F)$ is the (Fredholm) linearization of P at u, then $T_u\boldsymbol{\mathcal{M}} = \mathrm{Ker}(\mathcal{L}_u P)$ and $O_u\boldsymbol{\mathcal{M}} = \mathrm{Coker}(\mathcal{L}_u P)$.

Derived Differential Geometry

Lecture 3 of 14: C^∞-Algebraic Geometry

Dominic Joyce, Oxford University
Summer 2015

These slides, and references, etc., available at
`http://people.maths.ox.ac.uk/~joyce/DDG2015`

Plan of talk:

3. C^∞-Algebraic Geometry

Our goal is to define the 2-category of d-manifolds **dMan**.
First consider an algebro-geometric version of what we want to do.
A good algebraic analogue of smooth manifolds are *complex algebraic manifolds*, that is, separated smooth $\mathbb{C}$-schemes S of pure dimension. These form a full subcategory $\mathbf{AlgMan}_{\mathbb{C}}$ in the category $\mathbf{Sch}_{\mathbb{C}}$ of $\mathbb{C}$-schemes, and can roughly be characterized as the (sufficiently nice) objects S in $\mathbf{Sch}_{\mathbb{C}}$ whose cotangent complex $\mathbb{L}_S$ is a vector bundle (i.e. perfect in the interval $[0, 0]$).

To make a derived version of this, we first define an ∞-category $\mathbf{DerSch}_{\mathbb{C}}$ of *derived* $\mathbb{C}$*-schemes*, and then define the ∞-category $\mathbf{DerAlgMan}_{\mathbb{C}}$ of *derived complex algebraic manifolds* to be the full ∞-subcategory of objects $\mathbf{S}$ in $\mathbf{DerSch}_{\mathbb{C}}$ which are *quasi-smooth* (have cotangent complex $\mathbb{L}_S$ perfect in the interval $[-1, 0]$), and satisfy some other niceness conditions (separated, etc.).

Thus, we have 'classical' categories $\mathbf{AlgMan}_{\mathbb{C}} \subset \mathbf{Sch}_{\mathbb{C}}$, and related 'derived' ∞-categories $\mathbf{DerAlgMan}_{\mathbb{C}} \subset \mathbf{DerSch}_{\mathbb{C}}$.
David Spivak, a student of Jacob Lurie, defined an ∞-category $\mathbf{DerMan_{Spi}}$ of 'derived smooth manifolds' using a similar structure: he considered 'classical' categories $\mathbf{Man} \subset \mathbf{C^\infty Sch}$ and related 'derived' ∞-categories $\mathbf{DerMan_{Spi}} \subset \mathbf{DerC^\infty Sch}$. Here $\mathbf{C^\infty Sch}$ is C^∞*-schemes*, and $\mathbf{DerC^\infty Sch}$ *derived* C^∞*-schemes*. That is, before we can 'derive', we must first embed **Man** into a larger category of C^∞-schemes, singular generalizations of manifolds.
Our set-up is a simplification of Spivak's. I consider 'classical' categories $\mathbf{Man} \subset \mathbf{C^\infty Sch}$ and related 'derived' 2-categories $\mathbf{dMan} \subset \mathbf{dSpa}$, where **dMan** is *d-manifolds*, and **dSpa** *d-spaces*.
Here $\mathbf{dMan}, \mathbf{dSpa}$ are roughly 2-category truncations of Spivak's $\mathbf{DerMan}, \mathbf{DerC^\infty Sch}$ — see Borisov arXiv:1212.1153.
This lecture will introduce classical C^∞-schemes.

3.1. C^∞-rings

Algebraic geometry (based on algebra and polynomials) has excellent tools for studying singular spaces – the theory of schemes.
In contrast, conventional differential geometry (based on smooth real functions and calculus) deals well with nonsingular spaces – manifolds – but poorly with singular spaces.
There is a little-known theory of schemes in differential geometry, C^∞-*schemes*, going back to Lawvere, Dubuc, Moerdijk and Reyes, ... in synthetic differential geometry in the 1960s-1980s.
C^∞-schemes are essentially *algebraic* objects, on which smooth real functions and calculus make sense.
The theory works by replacing commutative rings or $\mathbb{K}$-algebras in algebraic geometry by algebraic objects called C^∞-*rings*.

Definition 3.1 (First definition of C^∞-ring)

A C^∞-*ring* is a set $\mathfrak{C}$ together with n-fold operations $\Phi_f : \mathfrak{C}^n \to \mathfrak{C}$ for all smooth maps $f : \mathbb{R}^n \to \mathbb{R}$, $n \geqslant 0$, satisfying:
Let $m, n \geqslant 0$, and $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$ and $g : \mathbb{R}^m \to \mathbb{R}$ be smooth functions. Define $h : \mathbb{R}^n \to \mathbb{R}$ by

$$h(x_1, \ldots, x_n) = g(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1 \ldots, x_n)),$$

for $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Then for all $c_1, \ldots, c_n$ in $\mathfrak{C}$ we have

$$\Phi_h(c_1, \ldots, c_n) = \Phi_g(\Phi_{f_1}(c_1, \ldots, c_n), \ldots, \Phi_{f_m}(c_1, \ldots, c_n)).$$

Also defining $\pi_j : (x_1, \ldots, x_n) \mapsto x_j$ for $j = 1, \ldots, n$ we have $\Phi_{\pi_j} : (c_1, \ldots, c_n) \mapsto c_j$.
A *morphism* of C^∞-rings is a map of sets $\phi : \mathfrak{C} \to \mathfrak{D}$ with $\Phi_f \circ \phi^n = \phi \circ \Phi_f : \mathfrak{C}^n \to \mathfrak{D}$ for all smooth $f : \mathbb{R}^n \to \mathbb{R}$. Write $\mathbf{C^\infty Rings}$ for the category of C^∞-rings.

Definition 3.2 (Second definition of C^∞-ring)

Write **Euc** for the full subcategory of manifolds **Man** with objects $\mathbb{R}^n$ for $n=0,1,\ldots$. That is, **Euc** is the category with objects Euclidean spaces $\mathbb{R}^n$, and morphisms smooth maps $f:\mathbb{R}^m\to\mathbb{R}^n$.
A *C^∞-ring* is a product-preserving functor $F:\textbf{Euc}\to\textbf{Sets}$. That is, F is a functor with functorial identifications
$F(\mathbb{R}^{m+n})=F(\mathbb{R}^m\times\mathbb{R}^n)\cong F(\mathbb{R}^m)\times F(\mathbb{R}^n)$ for all $m,n\geqslant 0$.
A morphism $\phi:F\to G$ of C^∞-rings F,G is a natural transformation of functors $\phi:F\Rightarrow G$.

Definitions 3.1 and 3.2 are equivalent as follows. Given $F:\textbf{Euc}\to\textbf{Sets}$ as above, define a set $\mathfrak{C}=F(\mathbb{R})$. As F is product-preserving, $F(\mathbb{R}^n)\cong F(\mathbb{R})^n=\mathfrak{C}^n$ for all $n\geqslant 0$. If $f:\mathbb{R}^n\to\mathbb{R}$ is smooth then $F(f):F(\mathbb{R}^n)\to F(\mathbb{R})$ is identified with a map $\Phi_f:\mathfrak{C}^n\to\mathfrak{C}$. Then $\bigl(\mathfrak{C},\Phi_{f,\,f:\mathbb{R}^n\to\mathbb{R}\;C^\infty}\bigr)$ is a C^∞-ring as in Definition 3.1. Conversely, given $\mathfrak{C}$ we define F with $F(\mathbb{R}^n)=\mathfrak{C}^n$.

Manifolds as C^∞-rings

Let X be a manifold, and write $\mathfrak{C}=C^\infty(X)$ for the set of smooth functions $c:X\to\mathbb{R}$. Let $f:\mathbb{R}^n\to\mathbb{R}$ be smooth. Define $\Phi_f:C^\infty(X)^n\to C^\infty(X)$ by $\Phi_f(c_1,\ldots,c_n)(x)=f\bigl(c_1(x),\ldots,c_n(x)\bigr)$ for $x\in X$. These make $C^\infty(X)$ into a C^∞-ring as in Definition 3.1.
Define $F:\textbf{Euc}\to\textbf{Sets}$ by $F(\mathbb{R}^n)=\mathrm{Hom}_{\textbf{Man}}(X,\mathbb{R}^n)$ and $F(f)=f\circ:\mathrm{Hom}_{\textbf{Man}}(X,\mathbb{R}^m)\to\mathrm{Hom}_{\textbf{Man}}(X,\mathbb{R}^n)$ for $f:\mathbb{R}^m\to\mathbb{R}^n$ smooth. Then F is a C^∞-ring as in Definition 3.2.
If $f:X\to Y$ is smooth map of manifolds then $f^*:C^\infty(Y)\to C^\infty(X)$ is a morphism of C^∞-rings; conversely, if $\phi:C^\infty(Y)\to C^\infty(X)$ is a morphism of C^∞-rings then $\phi=f^*$ for some unique smooth $f:X\to Y$. This gives a *full and faithful functor* $F:\textbf{Man}\to\mathbf{C^\infty Rings}^{\mathrm{op}}$ by $F:X\mapsto C^\infty(X)$, $F:f\mapsto f^*$.
Thus, we can think of manifolds as examples of C^∞-rings. But there are many more C^∞-rings than manifolds. For example, $C^0(X)$ is a C^∞-ring for any topological space X.

C^∞-rings as $\mathbb{R}$-algebras, ideals, and quotient C^∞-rings

Every C^∞-ring $\mathfrak{C}$ is a commutative $\mathbb{R}$-algebra, where addition is $c + d = \Phi_f(c,d)$ for $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x + y$, and multiplication is $c \cdot d = \Phi_g(c,d)$ for $g : \mathbb{R}^2 \to \mathbb{R}$, $g(x,y) = xy$, multiplication by $\alpha \in \mathbb{R}$ is $\alpha\, c = \Phi_h(c)$ for $h : \mathbb{R} \to \mathbb{R}$, $h(x) = \alpha x$. An ideal $I \subseteq \mathfrak{C}$ in a C^∞-ring $\mathfrak{C}$ is an ideal in $\mathfrak{C}$ as an $\mathbb{R}$-algebra. Then the quotient vector space $\mathfrak{C}/I$ is a commutative $\mathbb{R}$-algebra.

Proposition 3.3

If $\mathfrak{C}$ is a C^∞-ring and $I \subseteq \mathfrak{C}$ an ideal, then there is a unique C^∞-ring structure on $\mathfrak{C}/I$ such that the projection $\pi : \mathfrak{C} \to \mathfrak{C}/I$ is a morphism of C^∞-rings.

Definition

A C^∞-ring $\mathfrak{C}$ is called *finitely generated* if $\mathfrak{C} \cong C^\infty(\mathbb{R}^n)/I$ for some ideal $I \subseteq C^\infty(\mathbb{R}^n)$.

Proof of Proposition 3.3

Let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth, and $c_1 + I, \ldots, c_n + I \in \mathfrak{C}/I$. For $\pi : \mathfrak{C} \to \mathfrak{C}/I$ to be a morphism of C^∞-rings, we are forced to set

$$\Phi_f(c_1 + I, \ldots, c_n + I) = \Phi_f(c_1, \ldots, c_n) + I,$$

which determines the C^∞-ring structure on $\mathfrak{C}/I$ completely. The only thing to prove is that this is well-defined. That is, if $c_1', \ldots, c_n' \in \mathfrak{C}$ with $c_i - c_i' \in I$, so that $c_1 + I = c_1' + I, \ldots, c_n + I = c_n' + I$, we must show that

$$\Phi_f(c_1, \ldots, c_n) - \Phi_f(c_1', \ldots, c_n') \in I.$$

Proof of Proposition 3.3

Lemma 3.4 (Hadamard's Lemma)

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is smooth. Then there exist smooth $g_i : \mathbb{R}^{2n} \to \mathbb{R}$ for $i = 1, \ldots, n$ such that for all x_j, y_j we have

$$f(x_1, \ldots, x_n) - f(y_1, \ldots, y_n) = \sum_{i=1}^{n} g_i(x_1, \ldots, x_n, y_1, \ldots, y_n) \cdot (x_i - y_i).$$

Note that $g_i(x_1, \ldots, x_n, x_1, \ldots, x_n) = \frac{\partial f}{\partial x_i}(x_1, \ldots, x_n)$, so Hadamard's Lemma gives an algebraic interpretation of partial derivatives. The definition of C^∞-ring implies that

$$\Phi_f(c_1, \ldots, c_n) - \Phi_f(c'_1, \ldots, c'_n) = \sum_{i=1}^{n} \Phi_{g_i}(c_1, \ldots, c_n, c'_1, \ldots, c'_n) \cdot (c_i - c'_i),$$

which lies in I as $c_i - c'_i \in I$, as we have to prove.

Example 3.5 (Finitely presented C^∞-rings. Compare Example 1.1.)

Suppose $p_1, \ldots, p_k : \mathbb{R}^n \to \mathbb{R}$ are smooth functions. Then $C^\infty(\mathbb{R}^n)$ is a C^∞-ring, and so an $\mathbb{R}$-algebra. Write $I = (p_1, \ldots, p_k)$ for the ideal in $C^\infty(\mathbb{R}^n)$ generated by $p_1, \ldots, p_k$. Then $C^\infty(\mathbb{R}^n)/(p_1, \ldots, p_k)$ is a C^∞-ring, by Proposition 3.3. We think of it as the C^∞-ring of functions on the smooth space $X = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : p_i(x_1, \ldots, x_n) = 0,\ i = 1, \ldots, k\}$. Note that X may be singular.

Example 3.6

Let $I \subset C^\infty(\mathbb{R})$ be the ideal of all smooth $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = 0$ for all $x \geqslant 0$. Then I is *not finitely generated*, so $C^\infty(\mathbb{R})$ is *not noetherian* as an $\mathbb{R}$-algebra. This is one way in which C^∞-algebraic geometry behaves worse than ordinary algebraic geometry. We think of $C^\infty(\mathbb{R})/I$ as the C^∞-ring of smooth functions $f : [0, \infty) \to \mathbb{R}$.

Definition

A C^∞-ring $\mathfrak{C}$ is a C^∞-*local ring* if as an $\mathbb{R}$-algebra, $\mathfrak{C}$ has a unique maximal ideal $\mathfrak{m}$, with $\mathfrak{C}/\mathfrak{m} \cong \mathbb{R}$.

Example 3.7

Let X be a manifold, and $x \in X$. Write $C^\infty_x(X)$ for the C^∞-ring of germs of smooth functions $f : X \to \mathbb{R}$ at x. That is, elements of $C^\infty_x(X)$ are $\sim$-equivalence classes $[U, f]$ of pairs (U, f), where $x \in U \subseteq X$ is open and $f : U \to \mathbb{R}$ is smooth, and $(U, f) \sim (U', f')$ if there exists open $x \in U'' \subseteq U \cap U'$ with $f|_{U''} = f'|_{U''}$. Then $C^\infty_x(X)$ is a C^∞-local ring.

Definition

An ideal $I \subseteq C^\infty(\mathbb{R}^n)$ is called *fair* if for $f \in C^\infty(\mathbb{R}^n)$, $\pi_x(f) \in \pi_x(I)$ for all $x \in \mathbb{R}^n$ implies that $f \in I$, where $\pi_x : C^\infty(\mathbb{R}^n) \to C^\infty_x(\mathbb{R}^n)$ is the projection. A C^∞-ring $\mathfrak{C}$ is called *fair* if $\mathfrak{C} \cong C^\infty(\mathbb{R}^n)/I$ for $I \subseteq C^\infty(\mathbb{R}^n)$ a fair ideal.

Modules over C^∞-rings

Definition

Let $\mathfrak{C}$ be a C^∞-ring. A *module* over $\mathfrak{C}$ is a module over $\mathfrak{C}$ as an $\mathbb{R}$-algebra.

You might expect that the definition of module should involve the operations Φ_f as well as the $\mathbb{R}$-algebra structure, but it does not.

Example 3.8

Let X be a manifold, and $E \to X$ a vector bundle. Then $C^\infty(X)$ is a C^∞-ring, and the vector space $C^\infty(E)$ of smooth sections of E is a module over $C^\infty(X)$.

Cotangent modules

Definition

Let $\mathfrak{C}$ be a C^∞-ring, and M a $\mathfrak{C}$-module. A C^∞-*derivation* is an $\mathbb{R}$-linear map $\mathrm{d}:\mathfrak{C}\to M$ such that whenever $f:\mathbb{R}^n\to\mathbb{R}$ is a smooth map and $c_1,\ldots,c_n\in\mathfrak{C}$, we have

$$\mathrm{d}\Phi_f(c_1,\ldots,c_n)=\textstyle\sum_{i=1}^n \Phi_{\frac{\partial f}{\partial x_i}}(c_1,\ldots,c_n)\cdot \mathrm{d}c_i.$$

Note that d is *not* a morphism of $\mathfrak{C}$-modules. We call such a pair $\Omega_{\mathfrak{C}},\mathrm{d}_{\mathfrak{C}}$ a *cotangent module* for $\mathfrak{C}$ if it has the universal property that for any $\mathfrak{C}$-module M and C^∞-derivation $\mathrm{d}:\mathfrak{C}\to M$, there is a unique morphism of $\mathfrak{C}$-modules $\phi:\Omega_{\mathfrak{C}}\to M$ with $\mathrm{d}=\phi\circ\mathrm{d}_{\mathfrak{C}}$.

Every C^∞-ring has a cotangent module, unique up to isomorphism.

Example 3.9

Let X be a manifold, with cotangent bundle T^*X. Then $C^\infty(T^*X)$ is a cotangent module for the C^∞-ring $C^\infty(X)$.

3.2. Sheaves

Sheaves are a central idea in algebraic geometry.

Definition

Let X be a topological space. A *presheaf of sets* $\mathcal{E}$ on X consists of a set $\mathcal{E}(U)$ for each open $U\subseteq X$, and a restriction map $\rho_{UV}:\mathcal{E}(U)\to\mathcal{E}(V)$ for all open $V\subseteq U\subseteq X$, such that:

(i) $\mathcal{E}(\emptyset)=*$ is one point;
(ii) $\rho_{UU}=\mathrm{id}_{\mathcal{E}(U)}$ for all open $U\subseteq X$; and
(iii) $\rho_{UW}=\rho_{VW}\circ\rho_{UV}$ for all open $W\subseteq V\subseteq U\subseteq X$.

We call $\mathcal{E}$ a *sheaf* if also whenever $U\subseteq X$ is open and $\{V_i:i\in I\}$ is an open cover of U, then:

(iv) If $s,t\in\mathcal{E}(U)$ with $\rho_{UV_i}(s)=\rho_{UV_i}(t)$ for all $i\in I$, then $s=t$;
(v) If $s_i\in\mathcal{E}(V_i)$ for all $i\in I$ with $\rho_{V_i(V_i\cap V_j)}(s_i)=\rho_{V_j(V_i\cap V_j)}(s_j)$ in $\mathcal{E}(V_i\cap V_j)$ for all $i,j\in I$, then there exists $s\in\mathcal{E}(U)$ with $\rho_{UV_i}(s)=s_i$ for all $i\in I$. This s is unique by (iv).

Definition

Let $\mathcal{E},\mathcal{F}$ be (pre)sheaves on X. A *morphism* $\phi:\mathcal{E}\to\mathcal{F}$ consists of a map $\phi(U):\mathcal{E}(U)\to\mathcal{F}(U)$ for all open $U\subseteq X$, such that $\rho_{UV}\circ\phi(U)=\phi(V)\circ\rho_{UV}:\mathcal{E}(U)\to\mathcal{F}(V)$ for all open $V\subseteq U\subseteq X$. Then sheaves form a category.

If $\mathcal{C}$ is any category in which direct limits exist, such as the categories of sets, rings, vector spaces, C^∞-rings, ..., then we can define (pre)sheaves $\mathcal{E}$ of objects in $\mathcal{C}$ on X in the obvious way, and morphisms $\phi:\mathcal{E}\to\mathcal{F}$ by taking $\mathcal{E}(U)$ to be an object in $\mathcal{C}$, and $\rho_{UV}:\mathcal{E}(U)\to\mathcal{E}(V)$, $\phi(U):\mathcal{E}(U)\to\mathcal{F}(U)$ to be morphisms in $\mathcal{C}$, and $\mathcal{E}(\emptyset)$ to be a terminal object in $\mathcal{C}$ (e.g. the zero ring). So in particular, we can define *sheaves of C^∞-rings* on X.
Almost any class of functions on X, or sections of a bundle on X, will form a sheaf on X. To be a sheaf means to be 'local on X', determined by its behaviour on any cover of small open sets.

Stalks of sheaves

Definition

Let X be a topological space, and $\mathcal{E}$ a (pre)sheaf of sets (or C^∞-rings, etc.) on X, and $x\in X$. The *stalk* $\mathcal{E}_x$ of $\mathcal{E}$ at x is

$$\mathcal{E}_x=\varinjlim{}_{x\in U\subseteq X}\mathcal{E}(U),$$

where the direct limit (as a set, or C^∞-ring, etc.) is over all open $U\subseteq X$ with $x\in U$ using $\rho_{UV}:\mathcal{E}(U)\to\mathcal{E}(V)$ for open $x\in V\subseteq U\subseteq X$. That is, for all open $x\in U\subseteq X$ we have a morphism $\pi_x:\mathcal{E}(U)\to\mathcal{E}_x$, such that for all $x\in V\subseteq U\subseteq X$ we have $\pi_x=\pi_x\circ\rho_{UV}$, and $\mathcal{E}_x$ is universal with this property.

Example 3.10

Let X be a manifold. Define a sheaf of C^∞-rings $\mathcal{O}_X$ on X by $O_X(U)=C^\infty(U)$ for all open $U\subseteq X$, as a C^∞-ring, and $\rho_{UV}:C^\infty(U)\to C^\infty(V)$, $\rho_{UV}:f\mapsto f|_V$ for all open $V\subseteq U\subseteq X$. The stalk $\mathcal{O}_{X,x}$ at $x\in X$ is $C^\infty_x(X)$ from Example 3.7.

Sheafification and pullbacks

Definition

Let X be a topological space and $\mathcal{E}$ a presheaf (of sets, C^∞-rings, etc.) on X. A *sheafification* of $\mathcal{E}$ is a sheaf $\mathcal{E}'$ and a morphism of presheaves $\pi : \mathcal{E} \to \mathcal{E}'$, with the universal property that any morphism $\phi : \mathcal{E} \to \mathcal{F}$ with $\mathcal{F}$ a sheaf factorizes uniquely as $\phi = \phi' \circ \pi$ for $\phi' : \mathcal{E}' \to \mathcal{F}$.

Any presheaf $\mathcal{E}$ has a sheafification $\mathcal{E}'$, unique up to canonical isomorphism, and the stalks satisfy $\mathcal{E}_x \cong \mathcal{E}'_x$.

Definition

Let $f : X \to Y$ be a continuous map of topological spaces, and $\mathcal{E}$ a sheaf on Y. Define a presheaf $\mathcal{P}f^{-1}(\mathcal{E})$ on X by

$$\mathcal{P}f^{-1}(\mathcal{E}) = \varinjlim_{V \supseteq f(U)} \mathcal{E}(V),$$

where the direct limit is over open $V \subseteq Y$ with $f(U) \subseteq V$. Define the *pullback sheaf* $f^{-1}(\mathcal{E})$ to be the sheafification of $\mathcal{P}f^{-1}(\mathcal{E})$.

3.3. C^∞-schemes

We can now define C^∞-schemes almost exactly as for schemes in algebraic geometry, but replacing rings or $\mathbb{K}$-algebras by C^∞-rings.

Definition

A *C^∞-ringed space* $\underline{X} = (X, \mathcal{O}_X)$ is a topological space X with a sheaf of C^∞-rings $\mathcal{O}_X$. It is called a *local C^∞-ringed space* if the stalks $\mathcal{O}_{X,x}$ are C^∞-local rings for all $x \in X$.

A *morphism* $\underline{f} : \underline{X} \to \underline{Y}$ of C^∞-ringed spaces is $\underline{f} = (f, f^\sharp)$, where $f : X \to Y$ is a continuous map of topological spaces, and $f^\sharp : f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$ is a morphism of sheaves of C^∞-rings on X.

Write $\mathbf{C^\infty RS}$ for the category of C^∞-ringed spaces, and $\mathbf{LC^\infty RS}$ for the full subcategory of local C^∞-ringed spaces.

Definition

The *global sections functor* $\Gamma : \mathbf{LC^\infty RS} \to \mathbf{C^\infty Rings}^{\mathrm{op}}$ maps $\Gamma : (X, \mathcal{O}_X) \mapsto \mathcal{O}_X(X)$. It has a right adjoint, the *spectrum functor* $\mathrm{Spec} : \mathbf{C^\infty Rings}^{\mathrm{op}} \to \mathbf{LC^\infty RS}$. That is, for each C^∞-ring $\mathfrak{C}$ we construct a local C^∞-ringed space $\underline{X} = \mathrm{Spec}\,\mathfrak{C}$. Points $x \in X$ are $\mathbb{R}$-algebra morphisms $x : \mathfrak{C} \to \mathbb{R}$ (this implies x is a C^∞-ring morphism). Then each $c \in \mathfrak{C}$ defines a map $c : X \to \mathbb{R}$. We give X the weakest topology such that these $c : X \to \mathbb{R}$ are continuous for all $c \in \mathfrak{C}$. We don't use prime ideals.

In algebraic geometry, $\mathrm{Spec} : \mathbf{Rings^{op}} \to \mathbf{LRS}$ is full and faithful. In C^∞-algebraic geometry, it is full but not faithful, that is, Spec forgets some information, as we don't use prime ideals. But on the subcategory $\mathbf{C^\infty Rings^{fa}}$ of *fair* C^∞-rings, Spec is full and faithful.

Definition

A local C^∞-ringed space $\underline{X}$ is called an *affine C^∞-scheme* if $\underline{X} \cong \mathrm{Spec}\,\mathfrak{C}$ for some C^∞-ring $\mathfrak{C}$. We call $\underline{X}$ a *C^∞-scheme* if X can be covered by open subsets U with $(U, \mathcal{O}_X|_U)$ an affine C^∞-scheme. Write $\mathbf{C^\infty Sch}$ for the full subcategory of C^∞-schemes in $\mathbf{LC^\infty RS}$.

If X is a manifold, define a C^∞-scheme $\underline{X} = (X, \mathcal{O}_X)$ by $\mathcal{O}_X(U) = C^\infty(U)$ for all open $U \subseteq X$. Then $\underline{X} \cong \mathrm{Spec}\,C^\infty(X)$. This defines a full and faithful embedding $\mathbf{Man} \hookrightarrow \mathbf{C^\infty Sch}$. So we can regard manifolds as examples of C^∞-schemes.

Think of a C^∞-ringed space $\underline{X}$ as a topological space X with a notion of 'smooth function' $f : U \to \mathbb{R}$ for open $U \subseteq X$, i.e. $f \in \mathcal{O}_X(U)$. If $\underline{X}$ is a local C^∞-ringed space then the notion of 'value of f in $\mathbb{R}$ at a point $x \in U$' makes sense, since we can compose the maps $f \in \mathcal{O}_X(U) \xrightarrow{\pi_x} \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}/\mathfrak{m} \cong \mathbb{R}$. If $\underline{X}$ is a C^∞-scheme, then for small open $U \subseteq X$ we can locally reconstruct the sheaf $\mathcal{O}_X|_U$ from the C^∞-ring $\mathcal{O}_X(U)$.

All *fibre products* exist in $\mathbf{C^\infty Sch}$. In manifolds **Man**, fibre products $X \times_{g,Z,h} Y$ need exist only if $g : X \to Z$ and $h : Y \to Z$ are transverse. When g, h are not transverse, the fibre product $X \times_{g,Z,h} Y$ exists in $\mathbf{C^\infty Sch}$, but may not be a manifold.

We also define *vector bundles* and *quasicoherent sheaves* on a C^∞-scheme $\underline{X}$, as sheaves of $\mathcal{O}_X$-modules, and write $\mathrm{qcoh}(\underline{X})$ for the abelian category of quasicoherent sheaves. A C^∞-scheme $\underline{X}$ has a well-behaved *cotangent sheaf* $T^*\underline{X}$.

Differences with ordinary Algebraic Geometry

- In algebraic geometry, central examples of schemes such as $\mathbb{CP}^n$ are not affine. In C^∞-algebraic geometry, most interesting C^∞-schemes are affine (e.g. all manifolds), except for non-Hausdorff C^∞-schemes. But scheme theory is still useful, to glue things from local data.
- The topology on C^∞-schemes is finer than the Zariski topology on schemes – affine schemes are always Hausdorff. No need to introduce the étale topology.
- Can find smooth functions supported on (almost) any open set.
- (Almost) any open cover has a subordinate partition of unity.
- Our C^∞-rings $\mathfrak{C}$ are generally *not noetherian* as $\mathbb{R}$-algebras. So ideals I in $\mathfrak{C}$ may not be finitely generated, even in $C^\infty(\mathbb{R}^n)$. This means there is not a well-behaved notion of coherent sheaf.

Derived Differential Geometry

Lecture 4 of 14: 2-categories, d-spaces, and d-manifolds

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Summer 2015

These slides, and references, etc., available at
`http://people.maths.ox.ac.uk/~joyce/DDG2015`

Plan of talk:

4. 2-categories, d-spaces, and d-manifolds

Our goal is to define the 2-category of d-manifolds **dMan**. To do this we will define a 2-category **dSpa** of 'd-spaces', a kind of derived C^∞-scheme, and then define d-manifolds $\mathbf{dMan} \subset \mathbf{dSpa}$ to be a special kind of d-space, just as manifolds $\mathbf{Man} \subset \mathbf{C^\infty Sch}$ are a special kind of C^∞-scheme.
First we introduce 2-*categories*. There are two kinds, strict 2-categories and weak 2-categories. We will meet both, as d-manifolds and d-orbifolds $\mathbf{dMan}, \mathbf{dOrb}$ are strict 2-categories, but Kuranishi spaces **Kur** are a weak 2-category. Every weak 2-category $\mathcal{C}$ is equivalent as a weak 2-category to a strict 2-category $\mathcal{C}'$ (weak 2-categories can be 'strictified'), so there is no fundamental difference, but weak 2-categories have more notation.

4.1. 2-categories

A 2-category $\mathcal{C}$ has *objects* $X, Y, \ldots$, 1-*morphisms* $f, g : X \to Y$ (morphisms), and 2-*morphisms* $\eta : f \Rightarrow g$ (morphisms between morphisms). Here are some examples to bear in mind:

Example 4.1

(a) The strict 2-category $\mathfrak{Cat}$ has objects categories $\mathcal{C}, \mathscr{D}, \ldots,$ 1-morphisms functors $F, G : \mathcal{C} \to \mathscr{D}$, and 2-morphisms natural transformations $\eta : F \Rightarrow G$.
(b) The strict 2-category $\mathbf{Top^{ho}}$ of *topological spaces up to homotopy* has objects topological spaces $X, Y, \ldots$, 1-morphisms continuous maps $f, g : X \to Y$, and 2-morphisms isotopy classes $[H] : f \Rightarrow g$ of homotopies H from f to g. That is, $H : X \times [0, 1] \to Y$ is continuous with $H(x, 0) = f(x)$, $H(x, 1) = g(x)$, and $H, H' : X \times [0, 1] \to Y$ are isotopic if there exists continuous $I : X \times [0, 1]^2 \to Y$ with $I(x, s, 0) = H(x, s)$, $I(s, x, 1) = H'(x, s)$, $I(x, 0, t) = f(x)$, $I(x, 1, t) = g(x)$.

Definition

A (*strict*) 2-*category* $\mathcal{C}$ consists of a proper class of *objects* $\mathrm{Obj}(\mathcal{C})$, for all $X, Y \in \mathrm{Obj}(\mathcal{C})$ a category $\mathrm{Hom}(X, Y)$, for all X in $\mathrm{Obj}(\mathcal{C})$ an object id_X in $\mathrm{Hom}(X, X)$ called the *identity* 1-*morphism*, and for all X, Y, Z in $\mathrm{Obj}(\mathcal{C})$ a functor $\mu_{X,Y,Z} : \mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$. These must satisfy the *identity property*, that

$$\mu_{X,X,Y}(\mathrm{id}_X, -) = \mu_{X,Y,Y}(-, \mathrm{id}_Y) = \mathrm{id}_{\mathrm{Hom}(X,Y)} \tag{4.1}$$

as functors $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Y)$, and the *associativity property*, that

$$\mu_{W,Y,Z} \circ (\mu_{W,X,Y} \times \mathrm{id}) = \mu_{W,X,Z} \circ (\mathrm{id} \times \mu_{X,Y,Z}) \tag{4.2}$$

as functors $\mathrm{Hom}(W, X) \times \mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(W, X)$. Objects f of $\mathrm{Hom}(X, Y)$ are called 1-*morphisms*, written $f : X \to Y$. For 1-morphisms $f, g : X \to Y$, morphisms $\eta \in \mathrm{Hom}_{\mathrm{Hom}(X,Y)}(f, g)$ are called 2-*morphisms*, written $\eta : f \Rightarrow g$.

There are three kinds of composition in a 2-category, satisfying various associativity relations. If $f : X \to Y$ and $g : Y \to Z$ are 1-morphisms then $\mu_{X,Y,Z}(f, g)$ is the *horizontal composition of* 1-*morphisms*, written $g \circ f : X \to Z$. If $f, g, h : X \to Y$ are 1-morphisms and $\eta : f \Rightarrow g$, $\zeta : g \Rightarrow h$ are 2-morphisms then composition of η, ζ in $\mathrm{Hom}(X, Y)$ gives the *vertical composition of* 2-*morphisms* of η, ζ, written $\zeta \odot \eta : f \Rightarrow h$, as a diagram

$$\begin{tikzcd} X \arrow[r, bend left=60, "f"] \arrow[r, "g"] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3, "\Downarrow\eta"] \arrow[r, phantom, shift right=3, "\Downarrow\zeta"] & Y \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\zeta\odot\eta"] & Y. \end{tikzcd} \tag{4.3}$$

And if $f, \tilde f : X \to Y$ and $g, \tilde g : Y \to Z$ are 1-morphisms and $\eta : f \Rightarrow \tilde f$, $\zeta : g \Rightarrow \tilde g$ are 2-morphisms then $\mu_{X,Y,Z}(\eta,\zeta)$ is the *horizontal composition of 2-morphisms*, written $\zeta * \eta : g \circ f \Rightarrow \tilde g \circ \tilde f$, as a diagram

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "\tilde f"'] \arrow[r, phantom, "\Downarrow\eta"] & Y \arrow[r, bend left, "g"] \arrow[r, bend right, "\tilde g"'] \arrow[r, phantom, "\Downarrow\zeta"] & Z \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} X \arrow[r, bend left, "g\circ f"] \arrow[r, bend right, "\tilde g\circ\tilde f"'] \arrow[r, phantom, "\Downarrow\zeta*\eta"] & Z. \end{tikzcd} \tag{4.4}$$

There are also two kinds of identity: *identity 1-morphisms* $\mathrm{id}_X : X \to X$ and *identity 2-morphisms* $\mathrm{id}_f : f \Rightarrow f$.
A 2-morphism is a 2-*isomorphism* if it is invertible under vertical composition. A 2-category is called a (2,1)-*category* if all 2-morphisms are 2-isomorphisms. For example, stacks in algebraic geometry form a (2,1)-category.

In a 2-category $\mathfrak{C}$, there are three notions of when objects X, Y in $\mathfrak{C}$ are 'the same': *equality* $X = Y$, and *isomorphism*, that is we have 1-morphisms $f : X \to Y$, $g : Y \to X$ with $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$, and *equivalence*, that is we have 1-morphisms $f : X \to Y$, $g : Y \to X$ and 2-isomorphisms $\eta : g \circ f \Rightarrow \mathrm{id}_X$ and $\zeta : f \circ g \Rightarrow \mathrm{id}_Y$. Usually equivalence is the correct notion.
Commutative diagrams in 2-categories should in general only commute *up to (specified) 2-isomorphisms*, rather than strictly. A simple example of a commutative diagram in a 2-category $\mathfrak{C}$ is

$$\begin{tikzcd} & Y \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow\eta"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & {} & Z, \end{tikzcd}$$

which means that X, Y, Z are objects of $\mathfrak{C}$, $f : X \to Y$, $g : Y \to Z$ and $h : X \to Z$ are 1-morphisms in $\mathfrak{C}$, and $\eta : g \circ f \Rightarrow h$ is a 2-isomorphism.

Definition (Fibre products in 2-categories. Compare §2.3.)

Let $\mathcal{C}$ be a strict 2-category and $g : X \to Z$, $h : Y \to Z$ be 1-morphisms in $\mathcal{C}$. A *fibre product* $X \times_Z Y$ in $\mathcal{C}$ is an object W, 1-morphisms $\pi_X : W \to X$ and $\pi_Y : W \to Y$ and a 2-isomorphism $\eta : g \circ \pi_X \Rightarrow h \circ \pi_Y$ in $\mathcal{C}$ with the following universal property: suppose $\pi'_X : W' \to X$ and $\pi'_Y : W' \to Y$ are 1-morphisms and $\eta' : g \circ \pi'_X \Rightarrow h \circ \pi'_Y$ is a 2-isomorphism in $\mathcal{C}$. Then there exists a 1-morphism $b : W' \to W$ and 2-isomorphisms $\zeta_X : \pi_X \circ b \Rightarrow \pi'_X$, $\zeta_Y : \pi_Y \circ b \Rightarrow \pi'_Y$ such that the following diagram commutes:

$$\begin{tikzcd} g \circ \pi_X \circ b \arrow[r, Rightarrow, "\eta * \mathrm{id}_b"'] \arrow[d, Rightarrow, "\mathrm{id}_g * \zeta_X"'] & h \circ \pi_Y \circ b \arrow[d, Rightarrow, "\mathrm{id}_h * \zeta_Y"] \\ g \circ \pi'_X \arrow[r, Rightarrow, "\eta'"] & h \circ \pi'_Y. \end{tikzcd}$$

Furthermore, if $\tilde{b}, \tilde{\zeta}_X, \tilde{\zeta}_Y$ are alternative choices of b, ζ_X, ζ_Y then there should exist a unique 2-isomorphism $\theta : \tilde{b} \Rightarrow b$ with

$$\tilde{\zeta}_X = \zeta_X \odot (\mathrm{id}_{\pi_X} * \theta) \quad \text{and} \quad \tilde{\zeta}_Y = \zeta_Y \odot (\mathrm{id}_{\pi_Y} * \theta).$$

If a fibre product $X \times_Z Y$ exists, it is unique up to equivalence.

Weak 2-categories

A *weak 2-category*, or *bicategory*, is like a strict 2-category, except that the equations of functors (4.1), (4.2) are required to hold not up to equality, but up to specified natural isomorphisms. That is, a weak 2-category $\mathcal{C}$ consists of data $\mathrm{Obj}(\mathcal{C}), \mathrm{Hom}(X, Y), \mu_{X,Y,Z}$, id_X as above, but in place of (4.1), a natural isomorphism

$$\alpha : \mu_{W,Y,Z} \circ (\mu_{W,X,Y} \times \mathrm{id}) \Longrightarrow \mu_{W,X,Z} \circ (\mathrm{id} \times \mu_{X,Y,Z}),$$

and in place of (4.2), natural isomorphisms

$$\beta : \mu_{X,X,Y}(\mathrm{id}_X, -) \Longrightarrow \mathrm{id}, \quad \gamma : \mu_{X,Y,Y}(-, \mathrm{id}_Y) \Longrightarrow \mathrm{id},$$

satisfying some identities. That is, composition of 1-morphisms is associative *only up to specified 2-isomorphisms*, so for 1-morphisms $e : W \to X$, $f : X \to Y$, $g : Y \to Z$ we have a 2-isomorphism

$$\alpha_{g,f,e} : (g \circ f) \circ e \Longrightarrow g \circ (f \circ e).$$

Similarly identities $\mathrm{id}_X, \mathrm{id}_Y$ work up to 2-isomorphism, so for each $f : X \to Y$ we have 2-isomorphisms

$$\beta_f : f \circ \mathrm{id}_X \Longrightarrow f, \qquad \gamma_f : \mathrm{id}_Y \circ f \Longrightarrow f.$$

4.2. Differential graded C^∞-rings

As in §2, to define derived $\mathbb{K}$-schemes, we replaced commutative $\mathbb{K}$-algebras by commutative differential graded $\mathbb{K}$-algebras (or simplicial $\mathbb{K}$-algebras). So, to define derived C^∞-schemes, we should replace C^∞-rings by *differential graded C^∞-rings* (or perhaps simplicial C^∞-rings, as in Spivak and Borisov–Noël).

Definition

A *differential graded C^∞-ring* (or *dg C^∞-ring*) $\mathfrak{C}^\bullet = (\mathfrak{C}^*, \mathrm{d})$ is a commutative differential graded $\mathbb{R}$-algebra $(\mathfrak{C}^*, \mathrm{d})$ in degrees $\leqslant 0$, as in §2.2, together with the structure $(\Phi_f)_{f:\mathbb{R}^n\to\mathbb{R}\ C^\infty}$ of a C^∞-ring on $\mathfrak{C}^0$, such that the $\mathbb{R}$-algebra structures on $\mathfrak{C}^0$ from the C^∞-ring and the cdga over $\mathbb{R}$ agree.
A *morphism* $\phi : \mathfrak{C}^\bullet \to \mathfrak{D}^\bullet$ of dg C^∞-rings is maps $\phi^k : \mathfrak{C}^k \to \mathfrak{D}^k$ for all $k \leqslant 0$, such that $(\phi^k)_{k\leqslant 0}$ is a morphism of cdgas over $\mathbb{R}$, and $\phi^0 : \mathfrak{C}^0 \to \mathfrak{D}^0$ is a morphism of C^∞-rings.

Then dg C^∞-rings form an (∞-)category $\mathbf{DGC^\infty Rings}$.
One could use dg C^∞-rings to define 'derived C^∞-schemes' and 'derived C^∞-stacks' as functors $F : \mathbf{DGC^\infty Rings} \to \mathbf{SSets}$. An alternative is to use *simplicial C^∞-rings* $\mathbf{SC^\infty Rings}$, as in Spivak 2008, Borisov–Noel 2011, and Borisov 2012.

Example 4.2 (Kuranishi neighbourhoods. Compare Example 2.1.)

Let V be a smooth manifold, and $E \to V$ a smooth real vector bundle of rank n, and $s : V \to E$ a smooth section. Define a dg C^∞-ring $\mathfrak{C}^\bullet$ as follows: take $\mathfrak{C}^0 = C^\infty(V)$, with its natural $\mathbb{R}$-algebra and C^∞-ring structures. Set $\mathfrak{C}^k = C^\infty(\Lambda^{-k}E^*)$ for $k = -1, -2, \ldots, -n$, and $\mathfrak{C}^k = 0$ for $k < -n$. The multiplication $\mathfrak{C}^k \times \mathfrak{C}^l \to \mathfrak{C}^{k+l}$ are multiplication by functions in $C^\infty(V)$ if $k = 0$ or $l = 0$, and wedge product $\wedge : \Lambda^{-k}E^* \times \Lambda^{-l}E^* \to \Lambda^{-k-l}E^*$ if $k, l < 0$. The differential $\mathrm{d} : \mathfrak{C}^k \to \mathfrak{C}^{k+1}$ is contraction with s, $s\cdot : \Lambda^{-k}E^* \to \Lambda^{-k-1}E^*$.

Square zero dg C^∞-rings

We will use only a special class of dg C^∞-rings called *square zero dg C^∞-rings*, which form a 2-category $\mathbf{SZC^\infty Rings}$.

Definition

A dg C^∞-ring $\mathfrak{C}^\bullet$ is *square zero* if $\mathfrak{C}^i = 0$ for $i < -1$ and $\mathfrak{C}^{-1} \cdot \mathrm{d}[\mathfrak{C}^{-1}] = 0$. Then $\mathfrak{C}$ is $\mathfrak{C}^{-1} \xrightarrow{\mathrm{d}} \mathfrak{C}^0$, and $\mathrm{d}[\mathfrak{C}^{-1}]$ is a square zero ideal in the (ordinary) C^∞-ring $\mathfrak{C}^0$, and $\mathfrak{C}^{-1}$ is a module over the 'classical' C^∞-ring $H^0(\mathfrak{C}^\bullet) = \mathfrak{C}^0/\mathrm{d}[\mathfrak{C}^{-1}]$.
A 1-*morphism* $\alpha^\bullet : \mathfrak{C}^\bullet \to \mathfrak{D}^\bullet$ in $\mathbf{SZC^\infty Rings}$ is maps $\alpha^0 : \mathfrak{C}^0 \to \mathfrak{D}^0$, $\alpha^{-1} : \mathfrak{C}^{-1} \to \mathfrak{D}^{-1}$ preserving all the structure.
Then $H^0(\alpha^\bullet) : H^0(\mathfrak{C}) \to H^0(\mathfrak{D})$ is a morphism of C^∞-rings.
For 1-morphisms $\alpha^\bullet, \beta^\bullet : \mathfrak{C}^\bullet \to \mathfrak{D}^\bullet$ a 2-*morphism* $\eta : \alpha^\bullet \Rightarrow \beta^\bullet$ is a linear $\eta : \mathfrak{C}^0 \to \mathfrak{D}^{-1}$ with $\beta^0 = \alpha^0 + \mathrm{d} \circ \eta$ and $\beta^{-1} = \alpha^{-1} + \eta \circ \mathrm{d}$.
There is an embedding of (2-)categories $\mathbf{C^\infty Rings} \subset \mathbf{SZC^\infty Rings}$ as the (2-)subcategory of $\mathfrak{C}^\bullet$ with $\mathfrak{C}^{-1} = 0$.

There is a truncation functor $T : \mathbf{DGC^\infty Rings} \to \mathbf{SZC^\infty Rings}$, where if $\mathfrak{C}^\bullet$ is a dg C^∞-ring, then $\mathfrak{D}^\bullet = T(\mathfrak{C}^\bullet)$ is the square zero C^∞-ring with

$$\mathfrak{D}^0 = \mathfrak{C}^0/[\mathrm{d}\mathfrak{C}^{-1}]^2, \quad \mathfrak{D}^{-1} = \mathfrak{C}^{-1}/[\mathrm{d}\mathfrak{C}^{-2} + (\mathrm{d}\mathfrak{C}^{-1}) \cdot \mathfrak{C}^{-1})].$$

Applied to Example 4.2 this gives:

Example 4.3 (Kuranishi neighbourhoods. Compare Example 4.2.)

Let V be a manifold, $E \to V$ a vector bundle, and $s : V \to E$ a smooth section. Associate a square zero dg C^∞-ring $\mathfrak{C}^{-1} \xrightarrow{\mathrm{d}} \mathfrak{C}^0$ to the 'Kuranishi neighbourhood' (V, E, s) by

$$\mathfrak{C}^0 = C^\infty(V)/I_s^2, \qquad \mathfrak{C}^{-1} = C^\infty(E^*)/I_s \cdot C^\infty(E^*),$$
$$\mathrm{d}(\epsilon + I_s \cdot C^\infty(E^*)) = \epsilon(s) + I_s^2,$$

where $I_s = C^\infty(E^*) \cdot s \subset C^\infty(V)$ is the ideal generated by s.

These will be the local models for d-manifolds.

Cotangent complexes in the 2-category setting

Let $\mathfrak{C}^\bullet$ be a square zero dg C^∞-ring. Define the *cotangent complex* $\mathbb{L}_{\mathfrak{C}}^{-1} \xrightarrow{\mathrm{d}_{\mathfrak{C}}} \mathbb{L}_{\mathfrak{C}}^0$ to be the 2-term complex of $H^0(\mathfrak{C}^\bullet)$-modules

$$\mathfrak{C}^{-1} \xrightarrow{\quad \mathrm{d}_{\mathrm{DR}} \circ \mathrm{d} \quad} \Omega_{\mathfrak{C}^0} \otimes_{\mathfrak{C}^0} H^0(\mathfrak{C}^\bullet),$$

regarded as an element of the 2-category of 2-term complexes of $H^0(\mathfrak{C}^\bullet)$-modules, with $\Omega_{\mathfrak{C}^0}$ the cotangent module of the C^∞-ring $\mathfrak{C}^0$, as in §3.1. Let $\alpha^\bullet, \beta^\bullet : \mathfrak{C}^\bullet \to \mathfrak{D}^\bullet$ be 1-morphisms and $\eta : \alpha^\bullet \Rightarrow \beta^\bullet$ a 2-morphism in $\mathbf{SZC^\infty Rings}$. Then $H^0(\alpha^\bullet) = H^0(\beta^\bullet)$, so we may regard $\mathfrak{D}^{-1}$ as an $H^0(\mathfrak{C}^\bullet)$-module. And $\eta : \mathfrak{C}^0 \to \mathfrak{D}^{-1}$ is a derivation, so it factors through an $H^0(\mathfrak{C}^\bullet)$-linear map $\hat\eta : \Omega_{\mathfrak{C}^0} \otimes_{\mathfrak{C}^0} H^0(\mathfrak{C}^\bullet) \to \mathfrak{D}^{-1}$. We have a diagram

$$\begin{tikzcd}
\mathbb{L}_{\mathfrak{C}}^{-1} \arrow[rr, "\mathrm{d}_{\mathfrak{C}}"] \arrow[d, shift right, "\mathbb{L}_\alpha^{-1}"'] \arrow[d, shift left, "\mathbb{L}_\beta^{-1}"] & & \mathbb{L}_{\mathfrak{C}}^0 \arrow[dll, "\hat\eta"'] \arrow[d, shift right, "\mathbb{L}_\alpha^{0}"'] \arrow[d, shift left, "\mathbb{L}_\beta^{0}"] \\
\mathbb{L}_{\mathfrak{D}}^{-1} \arrow[rr, "\mathrm{d}_{\mathfrak{D}}"] & & \mathbb{L}_{\mathfrak{D}}^0 .
\end{tikzcd}$$

So 1-morphisms induce morphisms, and 2-morphisms homotopies, of cotangent complexes.

4.3. D-spaces

D-spaces are our notion of derived C^∞-scheme:

Definition

A *d-space* $\mathbf{X}$ is a topological space X with a sheaf of square zero dg-C^∞-rings $\mathcal{O}_{\mathbf{X}}^\bullet = \mathcal{O}_X^{-1} \xrightarrow{\mathrm{d}} \mathcal{O}_{\mathbf{X}}^0$, such that $\underline{X} = (X, H^0(\mathcal{O}_{\mathbf{X}}^\bullet))$ and $(X, \mathcal{O}_{\mathbf{X}}^0)$ are C^∞-schemes, and $\mathcal{O}_X^{-1}$ is quasicoherent over $\underline{X}$. We call $\underline{X}$ the *underlying classical C^∞-scheme*.
We require that the topological space X should be Hausdorff and second countable, and the underlying classical C^∞-scheme $\underline{X}$ should be *locally fair*, i.e. covered by open $\operatorname{Spec}\mathfrak{C} \cong \underline{U} \subseteq \underline{X}$ for $\mathfrak{C}$ a fair C^∞-ring. Basically this means $\underline{X}$ is locally finite-dimensional.

Note that $\mathcal{O}_{\mathbf{X}}^\bullet$ is an ordinary (strict) sheaf of square zero dg C^∞-rings, using only the objects and 1-morphisms in $\mathbf{SZC^\infty Rings}$, and not (as usual in DAG) a homotopy sheaf using 2-isomorphisms $\rho_{VW} \circ \rho_{UV} \Rightarrow \rho_{UW}$ for open $W \subseteq V \subseteq U \subseteq X$.

Definition

A 1-*morphism* $\mathbf{f}:\mathbf{X}\to\mathbf{Y}$ of d-spaces $\mathbf{X},\mathbf{Y}$ is $\mathbf{f}=(f,f^\sharp)$, where $f:X\to Y$ is a continuous map of topological spaces, and $f^\sharp:f^{-1}(\mathcal{O}^\bullet_{\mathbf{Y}})\to\mathcal{O}^\bullet_{\mathbf{X}}$ is a morphism of sheaves of square zero dg C^∞-rings on X. Then $\underline{f}=(f,H^0(f^\sharp)):\underline{X}\to\underline{Y}$ is a morphism of the underlying classical C^∞-schemes.

Definition

Let $\mathbf{f},\mathbf{g}:\mathbf{X}\to\mathbf{Y}$ be 1-morphisms of d-spaces, and suppose the continuous maps $f,g:X\to Y$ are equal. We have morphisms $f^\sharp,g^\sharp:f^{-1}(\mathcal{O}^\bullet_{\mathbf{Y}})\to\mathcal{O}^\bullet_{\mathbf{X}}$ of sheaves of square zero dg C^∞-rings. That is, $f^\sharp,g^\sharp$ are sheaves on X of 1-morphisms in $\mathbf{SZC^\infty Rings}$. A 2-*morphism* $\eta:\mathbf{f}\Rightarrow\mathbf{g}$ is a sheaf on X of 2-morphisms $\eta:f^\sharp\Rightarrow g^\sharp$ in $\mathbf{SZC^\infty Rings}$. That is, for each open $U\subseteq X$, we have a 2-morphism $\eta(U):f^\sharp(U)\Rightarrow g^\sharp(U)$ in $\mathbf{SZC^\infty Rings}$, with $\mathrm{id}_{\rho_{UV}}*\eta(U)=\eta(V)*\mathrm{id}_{\rho_{UV}}$ for all open $V\subseteq U\subseteq X$.

With the obvious notions of composition of 1- and 2-morphisms, and identities, d-spaces form a strict 2-category $\mathbf{dSpa}$, in which all 2-morphisms are 2-isomorphisms.
C^∞-schemes include into d-spaces as those $\mathbf{X}$ with $\mathcal{O}_X^{-1}=0$. Thus we have inclusions of (2-)categories $\mathbf{Man}\subset\mathbf{C^\infty Sch}\subset\mathbf{dSpa}$, so manifolds are examples of d-spaces.
The *cotangent complex* $\mathbb{L}^\bullet_{\mathbf{X}}$ of $\mathbf{X}$ is the sheaf of cotangent complexes of $\mathcal{O}^\bullet_{\mathbf{X}}$, a 2-term complex $\mathbb{L}^{-1}_{\mathbf{X}}\xrightarrow{\mathrm{d}_{\mathbf{X}}}\mathbb{L}^0_{\mathbf{X}}$ of quasicoherent sheaves on $\underline{X}$. Such complexes form a 2-category $\mathrm{qcoh}^{[-1,0]}(\underline{X})$.

Theorem 4.4

All fibre products exist in the 2-category $\mathbf{dSpa}$.

The proof is by construction: given 1-morphisms $\mathbf{g}:\mathbf{X}\to\mathbf{Z}$ and $\mathbf{h}:\mathbf{Y}\to\mathbf{Z}$, we write down an explicit d-space $\mathbf{W}$, 1-morphisms $\mathbf{e}:\mathbf{W}\to\mathbf{X}$, $\mathbf{f}:\mathbf{W}\to\mathbf{Y}$ and 2-isomorphism $\eta:\mathbf{g}\circ\mathbf{e}\Rightarrow\mathbf{h}\circ\mathbf{f}$, and verify by hand that it satisfies the universal property in §4.1.

Gluing d-spaces by equivalences

Theorem 4.5

Let $\mathbf{X},\mathbf{Y}$ be d-spaces, $\emptyset\neq\mathbf{U}\subseteq\mathbf{X}$, $\emptyset\neq\mathbf{V}\subseteq\mathbf{Y}$ open d-subspaces, and $\mathbf{f}:\mathbf{U}\to\mathbf{V}$ an equivalence in the 2-category **dSpa**. *Suppose the topological space $Z=X\cup_{U=V}Y$ made by gluing X,Y using f is Hausdorff. Then there exist a d-space $\mathbf{Z}$, unique up to equivalence in* **dSpa**, *open $\hat{\mathbf{X}},\hat{\mathbf{Y}}\subseteq\mathbf{Z}$ with $\mathbf{Z}=\hat{\mathbf{X}}\cup\hat{\mathbf{Y}}$, equivalences $\mathbf{g}:\mathbf{X}\to\hat{\mathbf{X}}$ and $\mathbf{h}:\mathbf{Y}\to\hat{\mathbf{Y}}$, and a 2-morphism $\eta:\mathbf{g}|_{\mathbf{U}}\Rightarrow\mathbf{h}\circ\mathbf{f}$.*

The proof is again by explicit construction. First we glue the classical C^∞-schemes $\underline{X},\underline{Y}$ on $\underline{U}\subseteq\underline{X},\underline{V}\subseteq\underline{Y}$ by the isomorphism $\underline{f}:\underline{U}\to\underline{V}$ to get a C^∞-scheme $\underline{Z}$. The definition of $\mathbf{Z}$ involves choosing a smooth partition of unity on $\underline{Z}$ subordinate to the open cover $\{\underline{U},\underline{V}\}$. This is possible in the world of C^∞-schemes, but would not work in conventional (derived) algebraic geometry.

Theorem 4.6

Suppose I is an indexing set, and $<$ is a total order on I, and $\mathbf{X}_i$ for $i\in I$ are d-spaces, and for all $i<j$ in I we are given open d-subspaces $\mathbf{U}_{ij}\subseteq\mathbf{X}_i$, $\mathbf{U}_{ji}\subseteq\mathbf{X}_j$ and an equivalence $\mathbf{e}_{ij}:\mathbf{U}_{ij}\to\mathbf{U}_{ji}$, such that for all $i<j<k$ in I we have a 2-commutative diagram

$$\begin{tikzcd}[column sep=small] & \mathbf{U}_{ji}\cap\mathbf{U}_{jk} \arrow[dr, "\mathbf{e}_{jk}|_{\mathbf{U}_{ji}\cap\mathbf{U}_{jk}}"] \arrow[d, phantom, "\Downarrow\eta_{ijk}"] & \\ \mathbf{U}_{ij}\cap\mathbf{U}_{ik} \arrow[ur, "\mathbf{e}_{ij}|_{\mathbf{U}_{ij}\cap\mathbf{U}_{ik}}"] \arrow[rr, "\mathbf{e}_{ik}|_{\mathbf{U}_{ij}\cap\mathbf{U}_{ik}}"] & {} & \mathbf{U}_{ki}\cap\mathbf{U}_{kj}. \end{tikzcd} \tag{4.5}$$

Define the quotient topological space $Z=(\coprod_{i\in I}X_i)/\sim$, where $\sim$ is generated by $x_i\sim x_j$ if $i<j$, $x_i\in U_{ij}\subseteq X_i$ and $x_j\in U_{ji}\subseteq X_j$ with $e_{ij}(x_i)=x_j$. Suppose Z is Hausdorff and second countable. Then there exist a d-space $\mathbf{Z}$ and a 1-morphism $\mathbf{f}_i:\mathbf{X}_i\to\mathbf{Z}$ which is an equivalence with an open d-subspace $\hat{\mathbf{X}}_i\subseteq\mathbf{Z}$ for all $i\in I$, where $\mathbf{Z}=\bigcup_{i\in I}\hat{\mathbf{X}}_i$, such that $\mathbf{f}_i(\mathbf{U}_{ij})=\hat{\mathbf{X}}_i\cap\hat{\mathbf{X}}_j$ for $i<j$ in I, and there exists a 2-morphism $\zeta_{ij}:\mathbf{f}_j\circ\mathbf{e}_{ij}\Rightarrow\mathbf{f}_i|_{\mathbf{U}_{ij}}$. The d-space $\mathbf{Z}$ is unique up to equivalence, and is independent of choice of η_{ijk}.

Theorem 4.6 generalizes Theorem 4.5 to gluing many d-spaces by equivalences. It is important that the 2-isomorphisms η_{ijk} in (4.5) are only required to exist, they need not satisfy any conditions on quadruple overlaps, etc., and $\mathbf{Z}$ is independent of the choice of η_{ijk}. Because of this, Theorem 4.6 actually makes sense as a statement in the homotopy category $\mathrm{Ho}(\mathbf{dSpa})$. The analogue is false for gluing by equivalences for orbifolds **Orb**, d-orbifolds **dOrb**, and d-stacks **dSta**.

4.4. D-manifolds

Definition

A *d-manifold* $\mathbf{X}$ of *virtual dimension* $n \in \mathbb{Z}$ is a d-space $\mathbf{X}$ such that $\mathbf{X}$ is covered by open d-subspaces $\mathbf{Y} \subset \mathbf{X}$ with equivalences $\mathbf{Y} \simeq U \times_{g,W,h} V$, where U, V, W are manifolds with $\dim U + \dim V - \dim W = n$, regarded as d-spaces by $\mathbf{Man} \subset \mathbf{C^\infty Sch} \subset \mathbf{dSpa}$, and $g : U \to W$, $h : V \to W$ are smooth maps, and $U \times_{g,W,h} V$ is the fibre product in the 2-category **dSpa**. Write **dMan** for the full 2-subcategory of d-manifolds in **dSpa**.

Note that the fibre product $U \times_W V$ exists by Theorem 4.4, and must be taken in **dSpa** as a 2-category, not as an ordinary category Alternatively, we can write the local models as $\mathbf{Y} \simeq V \times_{0,E,s} V$, where V is a manifold, $E \to V$ a vector bundle, $s : V \to E$ a smooth section, and $n = \dim V - \operatorname{rank} E$. Then (V, E, s) is a *Kuranishi neighbourhood* on $\mathbf{X}$, as in Fukaya–Oh–Ohta–Ono.

Thus, a d-manifold $\mathbf{X}$ is a 'derived' geometric space covered by simple, differential-geometric local models: they are fibre products $U \times_{g,W,h} V$ for smooth maps of manifolds $g : U \to W$, $h : V \to W$, or they are the zeroes $s^{-1}(0)$ of a smooth section $s : V \to E$ of a vector bundle $E \to V$ over a manifold V.
However, as usual in derived geometry, the way in which these local models are glued together (by equivalences in the 2-category **dSpa**) is more mysterious, is weaker than isomorphisms, and takes some work to understand. We discuss this later in the course.
If $\mathbf{g} : \mathbf{X} \to \mathbf{Z}$, $\mathbf{h} : \mathbf{Y} \to \mathbf{Z}$ are 1-morphisms in **dMan**, then Theorem 4.4 says that a fibre product $\mathbf{W} = \mathbf{X} \times_{\mathbf{g},\mathbf{Z},\mathbf{h}} \mathbf{Y}$ exists in **dSpa**. If $\mathbf{W}$ is a d-manifold (which is a local question on $\mathbf{W}$) then $\mathbf{W}$ is also a fibre product in **dMan**. So we will give be able to give useful criteria for existence of fibre products in **dMan**.

Theorems 4.5 and 4.6 immediately lift to results on gluing by equivalences in **dMan**, taking $\mathbf{U}, \mathbf{V}, \mathbf{X}_i$ to be d-manifolds of a fixed virtual dimension $n \in \mathbb{Z}$. Thus, we can define d-manifolds by gluing together local models by equivalences. This is very useful, as natural examples (e.g. moduli spaces) are often presented in terms of local models somehow glued on overlaps.
I chose to use square zero dg C^∞-rings to define **dSpa**, **dMan** (rather than, say, general dg C^∞-rings) as they are very 'small' — they are essentially the minimal extension of classical C^∞-rings which remembers the 'derived' information I care about (in particular, sufficient to form virtual cycles for derived manifolds).
This has the advantage of making the theory simpler than it could have been, e.g. by using 2-categories rather than ∞-categories, whilst still having good properties, e.g. 'correct' fibre products and gluing by equivalences. A possible disadvantage is that they forget 'higher obstructions', which occur in some moduli problems.

Derived Differential Geometry

Lecture 5 of 14:
Differential-geometric description of d-manifolds

Dominic Joyce, Oxford University
Summer 2015

These slides, and references, etc., available at
`http://people.maths.ox.ac.uk/~joyce/DDG2015`

Plan of talk:

5. Differential-geometric description of d-manifolds

We have defined a strict 2-category **dSpa** of d-spaces $\mathbf{X} = (X, \mathcal{O}_X^\bullet)$, which are topological spaces X equipped with a sheaf of square zero dg C^∞-rings $\mathcal{O}_X^\bullet$. We have full (2-)subcategories $\mathbf{Man} \subset \mathbf{C^\infty Sch} \subset \mathbf{dSpa}$, so that we may regard manifolds as examples of d-spaces. All fibre products exist in **dSpa**.

A d-space $\mathbf{X}$ is called a *d-manifold* of *virtual dimension* $n \in \mathbb{Z}$ if it is locally modelled on fibre products $V \times_{0,E,s} V$ in **dSpa**, where V is a manifold, $E \to V$ a vector bundle with $\dim V - \operatorname{rank} E = n$, and $s : V \to E$ a smooth section. D-manifolds form a full 2-subcategory $\mathbf{dMan} \subset \mathbf{dSpa}$.

To actually do stuff with d-manifolds, it is very useful to be able to describe objects, 1-morphisms and 2-morphisms in **dMan** not using square zero dg C^∞-rings, but using honest differential-geometric objects: manifolds, vector bundles, sections, and smooth maps.

I will define a family of explicit 'standard model' d-manifolds $\mathbf{S}_{V,E,s}$, related to Example 4.3, depending on a manifold V, vector bundle $E \to V$ and section $s : V \to E$. We can describe 1-morphisms $\mathbf{f}, \mathbf{g} : \mathbf{S}_{V,E,s} \to \mathbf{S}_{W,F,t}$ and 2-morphisms $\eta : \mathbf{f} \Rightarrow \mathbf{g}$ completely in terms of the differential geometry of V, E, s, W, F, t. For this we will need '$O(s)$' and '$O(s^2)$' notation, defined in §5.1.

As every d-manifold $\mathbf{X}$ is locally equivalent to standard models $\mathbf{S}_{V,E,s}$, this enables us to describe d-manifolds and their 1- and 2-morphisms locally, solely in differential-geometric language.

In fact we can use these ideas to give an alternative definition of a (weak) 2-category of derived manifolds $\mathbf{Kur_{trG}}$ involving only manifolds and differential geometry, not using (dg) C^∞-rings and C^∞-schemes at all. This is the theory of (*M-*)*Kuranishi spaces*, and will be the subject of lectures 6-8.

5.1. The $O(s)$ and $O(s^2)$ notation

Definition

Let V be a manifold, $E \to V$ a vector bundle, and $s : V \to E$ be a smooth section of E, that is, $s \in C^\infty(E)$.

- If $f : V \to \mathbb{R}$ is smooth, we write '$f = O(s)$' if $f = \alpha \cdot s$ for some $\alpha \in C^\infty(E^*)$, and '$f = O(s^2)$' if $f = \beta \cdot (s \otimes s)$ for some $\beta \in C^\infty(E^* \otimes E^*)$.
- If $F \to V$ is a another vector bundle and $t \in C^\infty(F)$, we write '$t = O(s)$' if $t = \alpha \cdot s$ for some $\alpha \in C^\infty(F \otimes E^*)$, and '$t = O(s^2)$' if $t = \beta \cdot (s \otimes s)$ for some $\beta \in C^\infty(F \otimes E^* \otimes E^*)$.

In terms of the $\mathbb{R}$-algebra (or C^∞-ring) $C^\infty(V)$, $f = O(s)$ means $f \in I_s \subseteq C^\infty(V)$, and $f = O(s^2)$ means $f \in I_s^2 \subseteq C^\infty(V)$, where $I_s = C^\infty(E^*) \cdot s$ is the ideal in $C^\infty(V)$ generated by s. Similarly $t = O(s) \Leftrightarrow t \in I_s \cdot C^\infty(F)$ and $t = O(s^2) \Leftrightarrow t \in I_s^2 \cdot C^\infty(F)$.

Definition

Let V be a manifold, $E \to V$ a vector bundle, and $s \in C^\infty(E)$. Let W be another manifold, and $f, g : V \to W$ be smooth maps.

- We write '$g = f + O(s)$' if for all smooth $h : W \to \mathbb{R}$ we have $h \circ g - h \circ f = O(s)$ as smooth functions $V \to \mathbb{R}$.
- Similarly, we write '$g = f + O(s^2)$' if for all smooth $h : W \to \mathbb{R}$ we have $h \circ g - h \circ f = O(s^2)$.
- Let $v \in C^\infty(f^*(TW))$ with $v = O(s)$. Then we write '$g = f + v + O(s^2)$' if $h \circ g - f^*(\mathrm{d}h) \cdot v - h \circ f = O(s^2)$ for all smooth $h : W \to \mathbb{R}$, where $f^*(\mathrm{d}h)$ lies in $C^\infty(f^*(T^*W))$.

This is more tricky: note that f, g and v do not lie in the same vector space, so '$g - f - v$' does not make sense. Nonetheless $g = f + v + O(s^2)$ makes sense.
In terms of C^∞-schemes, $g = f + O(s)$ iff $g|_{\underline{X}} = f|_{\underline{X}}$, where $\underline{X} \subseteq V$ is the C^∞-subscheme defined by $s = 0$.

Definition

Let V be a manifold, $E \to V$ a vector bundle, and $s \in C^\infty(E)$. Let W be another manifold, and $f, g : V \to W$ be smooth maps with $g = f + O(s)$. Let $F \to W$ be a vector bundle, and $t \in C^\infty(f^*(F))$, $u \in C^\infty(g^*(F))$.
We say that '$u = t + O(s)$' if for all $\gamma \in C^\infty(F^*)$ we have $u \cdot g^*(\gamma) - t \cdot f^*(\gamma) = O(s)$ as smooth functions $V \to \mathbb{R}$.

Note that t, u are sections of different vector bundles, so '$u - t$' does not make sense. Nonetheless '$u = t + O(s)$' makes sense. In terms of C^∞-schemes, if $\underline{X} \subseteq V$ is the C^∞-subscheme defined by $s = 0$, then $g = f + O(s)$ implies that $g|_{\underline{X}} = f|_{\underline{X}}$, so $g^*(F)|_{\underline{X}}$ and $f^*(F)|_{\underline{X}}$ are the same vector bundle. Then $u = t + O(s)$ means that $u|_{\underline{X}} = t|_{\underline{X}}$ as sections of $g^*(F)|_{\underline{X}} = f^*(F)|_{\underline{X}}$.

5.2. Standard model d-manifolds

Proposition 5.1

Let $\mathbf{X}$ *be a d-manifold, with* $\mathrm{vdim}\,\mathbf{X} = n$. *Then the following are equivalent:*

(i) $\mathbf{X} \simeq U \times_{g,W,h} V$ *in* **dSpa**, *where* U, V, W *are manifolds,* $g : U \to W$, $h : V \to W$ *are smooth, and* $\dim U + \dim V - \dim W = n$.

(ii) $\mathbf{X} \simeq U \times_{i,W,j} V$ *in* **dSpa**, *where* W *is a manifold,* $U, V \subseteq W$ *are submanifolds with inclusions* $i : U \hookrightarrow W$, $j : V \hookrightarrow W$, *and* $\dim U + \dim V - \dim W = n$.

(iii) $\mathbf{X} \simeq V \times_{0,E,s} V$ *in* **dSpa**, *where* V *is a manifold,* $E \to V$ *is a vector bundle, and* $s \in C^\infty(E)$, *with* $\dim V - \mathrm{rank}\,E = n$.

We call $\mathbf{X}$ *satisfying* **(i)–(iii)** *a* **principal d-manifold**.

Every d-manifold $\mathbf{X}$ can be covered by open $\mathbf{Y} \subseteq \mathbf{X}$ with $\mathbf{Y}$ principal. We prefer to use model number (iii).

Definition

Let V be a manifold, $E \to V$ a vector bundle, and $s: V \to E$ a smooth section. As in Example 4.2, define a square zero dg C^∞-ring $\mathfrak{C}^\bullet = [\mathfrak{C}^{-1} \xrightarrow{\mathrm{d}} \mathfrak{C}^0]$ by

$$\mathfrak{C}^0 = C^\infty(V)/I_s^2, \qquad \mathfrak{C}^{-1} = C^\infty(E^*)/I_s \cdot C^\infty(E^*),$$
$$\mathrm{d}(\epsilon + I_s \cdot C^\infty(E^*)) = \epsilon(s) + I_s^2,$$

where $I_s = C^\infty(E^*) \cdot s \subset C^\infty(V)$ is the ideal generated by s. Define $\mathbf{S}_{V,E,s} = \mathbf{Spec}\,\mathfrak{C}^\bullet$. We call $\mathbf{S}_{V,E,s}$ a *standard model d-manifold*. It has topological space $S_{V,E,s} = s^{-1}(0) \subseteq V$.

Then $\mathbf{S}_{V,E,s} \simeq V \times_{0,E,s} V$, as in Proposition 5.1(iii). Now writing $\mathbf{S}_{V,E,s}$ as a fibre product only characterizes it up to equivalence in the 2-category **dSpa**. But writing $\mathbf{S}_{V,E,s} = \mathbf{Spec}\,\mathfrak{C}^\bullet$ characterizes it uniquely (at least, up to canonical 1-isomorphism) in **dSpa**. This will be important.

Open d-submanifolds of standard model d-manifolds

Let V, E, s be as above, and suppose $V' \subseteq V$ is open. Write $E' = E|_{V'}$ and $s' = s|_{V'}$. Then we have standard model d-manifolds $\mathbf{S}_{V,E,s}$ and $\mathbf{S}_{V',E',s'}$, with topological spaces $S_{V,E,s} = s^{-1}(0)$ and $S_{V',E',s'} = s^{-1}(0) \cap V'$.
In fact $\mathbf{S}_{V',E',s'} \subseteq \mathbf{S}_{V,E,s}$ is an open d-submanifold.
In particular, if V' is an open neighbourhood of $s^{-1}(0)$ in V, then $\mathbf{S}_{V',E',s'} = \mathbf{S}_{V,E,s}$. This means that we can always restrict to an arbitrarily small open neighbourhood of $s^{-1}(0)$ in V without changing anything; in effect, we can take germs about $s^{-1}(0)$ in V.
We have dg C^∞-rings $\mathfrak{C}^\bullet, \mathfrak{C}'^\bullet$ from (V, E, s) and (V', E', s'), and the natural restriction morphism $\iota : \mathfrak{C}^\bullet \to \mathfrak{C}'^\bullet$ is an isomorphism.

5.3. Standard model 1- and 2-morphisms

Definition

Let V, W be manifolds, $E \to V$, $F \to W$ be vector bundles, and $s \in C^\infty(E)$, $t \in C^\infty(F)$. Let V' be an open neighbourhood of $s^{-1}(0)$ in V, and $E' = E|_{V'}$, $s' = s|_{V'}$. Write $\mathfrak{C}^\bullet, \mathfrak{C}'^\bullet, \mathfrak{D}^\bullet$ for the square zero dg C^∞-rings from $(V, E, s), (V', E', s'), (W, F, t)$, so that $\mathbf{S}_{V,E,s} = \mathbf{Spec}\,\mathfrak{C}^\bullet$, $\mathbf{S}_{W,F,t} = \mathbf{Spec}\,\mathfrak{D}^\bullet$, and $\iota : \mathfrak{C}^\bullet \xrightarrow{\cong} \mathfrak{C}'^\bullet$.

Suppose $f : V' \to W$ is smooth, and $\hat{f} : E' \to f^*(F)$ is a morphism of vector bundles on V' with $\hat{f} \circ s' = f^*(t) + O(s^2)$ in $C^\infty(f^*(F))$. Define a morphism $\alpha : \mathfrak{D}^\bullet \to \mathfrak{C}'^\bullet$ of dg C^∞-rings by

$$\begin{array}{ccc} \mathfrak{D}^{-1} = C^\infty(F^*)/I_t \cdot C^\infty(F^*) & \xrightarrow[\mathrm{d}=t\cdot]{} & \mathfrak{D}^0 = C^\infty(W)/I_t^2 \\ \Big\downarrow{\scriptstyle \alpha^{-1}=\hat{f}^*\circ f^*} & & {\scriptstyle \alpha^0=f^*}\Big\downarrow \\ \mathfrak{C}'^{-1} = C^\infty(E'^*)/I_{s'} \cdot C^\infty(E'^*) & \xrightarrow{\mathrm{d}=s'\cdot} & \mathfrak{C}'^0 = C^\infty(V')/I_{s'}^2. \end{array}$$

Define $\mathbf{S}_{V',f,\hat{f}} = \mathbf{Spec}(\iota^{-1} \circ \alpha) : \mathbf{S}_{V,E,s} \to \mathbf{S}_{W,F,t}$. We call $\mathbf{S}_{V',f,\hat{f}}$ a *standard model 1-morphism*.

Theorem 5.2

Let $\mathbf{S}_{V,E,s}, \mathbf{S}_{W,F,t}$ be standard model d-manifolds. Then

(a) *Suppose $\mathbf{g} : \mathbf{S}_{V,E,s} \to \mathbf{S}_{W,F,t}$ is a 1-morphism in* **dMan**. *Then $\mathbf{g} = \mathbf{S}_{V',f,\hat{f}}$ for some standard model 1-morphism $\mathbf{S}_{V',f,\hat{f}}$ defined using $s^{-1}(0) \subseteq V' \subseteq V$, $f : V' \to W$, $\hat{f} : E' \to f^*(F)$.*

(b) *Suppose $\mathbf{S}_{V'_1,f_1,\hat{f}_1}, \mathbf{S}_{V'_2,f_2,\hat{f}_2} : \mathbf{S}_{V,E,s} \to \mathbf{S}_{W,F,t}$ are standard model 1-morphisms defined for $i=1,2$ using $s^{-1}(0) \subseteq V'_i \subseteq V$, $f_i : V'_i \to W$ and $\hat{f}_i : E'_i \to f_i^*(F)$. Then $\mathbf{S}_{V'_1,f_1,\hat{f}_1} = \mathbf{S}_{V'_2,f_2,\hat{f}_2}$ iff $f_2|_{V'_1 \cap V'_2} = f_1|_{V'_1 \cap V'_2} + O(s^2)$ and $\hat{f}_2|_{V'_1 \cap V'_2} = \hat{f}_1|_{V'_1 \cap V'_2} + O(s)$.*

Sketch proof.

For (a), we show $\mathbf{g} = \mathbf{Spec}\,\alpha$ for $\alpha : \mathfrak{D}^\bullet \to \mathfrak{C}^\bullet$ a morphism of dg C^∞-rings, and then show α is induced from some $V', f, \hat{f}$. For (b), we show the morphisms of dg C^∞-rings $\alpha_1, \alpha_2 : \mathfrak{D}^\bullet \to \mathfrak{C}^\bullet$ are equal iff $f_2 = f_1 + O(s^2)$ and $\hat{f}_2 = \hat{f}_1 + O(s)$. $\square$

From the point of view of 2-categories, Theorem 5.2 is a perverse result: we characterize the 1-morphisms $\mathbf{g} : \mathbf{S}_{V,E,s} \to \mathbf{S}_{W,F,t}$ completely as a set, *not* up to 2-isomorphism.
If we were to replace $\mathbf{S}_{V,E,s}, \mathbf{S}_{W,F,t}$ by equivalent objects in **dMan**, then the set of 1-morphisms $\mathbf{g} : \mathbf{S}_{V,E,s} \to \mathbf{S}_{W,F,t}$ might change (though the set of 2-isomorphism classes of 1-morphisms $\mathbf{g}$ would not), and Theorem 5.2 would be false.
The theorem depends upon using the particular model $\mathbf{S}_{V,E,s} = \mathbf{Spec}\,\mathfrak{C}^\bullet$ for the equivalence class of objects in **dSpa** representing the fibre product $V \times_{0,E,s} V$.
Next we need to understand 2-morphisms $\eta : \mathbf{S}_{V'_1,f_1,\hat f_1} \Rightarrow \mathbf{S}_{V'_2,f_2,\hat f_2}$ between standard model 1-morphisms.

'Standard model' 2-morphisms

Definition

Let $\mathbf{S}_{V,E,s}, \mathbf{S}_{W,F,t}$ be standard model d-manifolds, and $\mathbf{S}_{V'_1,f_1,\hat f_1}, \mathbf{S}_{V'_2,f_2,\hat f_2} : \mathbf{S}_{V,E,s} \to \mathbf{S}_{W,F,t}$ be standard model 1-morphisms. Suppose $f_2 = f_1 + O(s)$ on $V'' := V'_1 \cap V'_2 \subseteq V$. Let $\Lambda : E|_{V''} \to f_1|^*_{V''}(TW)$ be a vector bundle morphism, with

$$f_2 = f_1 + \Lambda \cdot s + O(s^2) \quad \text{and} \quad \hat f_2 = \hat f_1 + \Lambda \cdot f^*(\mathrm{d}t) + O(s). \quad (5.1)$$

Define the 'standard model' 2-morphism $\mathbf{S}_\Lambda : \mathbf{S}_{V'_1,f_1,\hat f_1} \Rightarrow \mathbf{S}_{V'_2,f_2,\hat f_2}$ to be **Spec** of the composition

$$\begin{matrix}\mathfrak{D}^0 = \\ C^\infty(W)/I_t^2\end{matrix} \xrightarrow{\Lambda^* \circ \mathrm{d}} \begin{matrix}\mathfrak{C}''^{-1} = \\ C^\infty(E''^*)/I_{s''} \cdot C^\infty(E''^*)\end{matrix} \xrightarrow{\iota''^{-1}} \mathfrak{C}^{-1},$$

where $\mathfrak{C}''^\bullet$ is from $(V'', E'' = E|_{V''}, s'' = s|_{V''})$ and $\iota'' : \mathfrak{C}^\bullet \to \mathfrak{C}''^\bullet$ the natural isomorphism.

Here is the analogue of Theorem 5.2, with a similar proof:

Theorem 5.3

Let $\mathbf{S}_{V,E,s}, \mathbf{S}_{W,F,t}$ *be standard model d-manifolds, and* $\mathbf{S}_{V_1',f_1,\hat f_1}, \mathbf{S}_{V_2',f_2,\hat f_2} : \mathbf{S}_{V,E,s} \to \mathbf{S}_{W,F,t}$ *be standard model 1-morphisms in* **dMan**. *Then*

(a) *Suppose* $\eta : \mathbf{S}_{V_1',f_1,\hat f_1} \Rightarrow \mathbf{S}_{V_2',f_2,\hat f_2}$ *is a 2-morphism in* **dMan**. *Then* $\eta = \mathbf{S}_\Lambda$ *for some standard model 2-morphism defined using* $\Lambda : E|_{V_1'\cap V_2'} \to f_1|^*_{V_1'\cap V_2'}(TW)$.

(b) *Suppose* $\mathbf{S}_{\Lambda_1}, \mathbf{S}_{\Lambda_2} : \mathbf{S}_{V_1',f_1,\hat f_1} \Rightarrow \mathbf{S}_{V_2',f_2,\hat f_2}$ *are standard model 2-morphisms. Then* $\mathbf{S}_{\Lambda_1} = \mathbf{S}_{\Lambda_2}$ *iff* $\Lambda_2 = \Lambda_1 + O(s)$.

Conclusions

Write **SMod** for the full 2-subcategory of **dMan** with objects standard model d-manifolds $\mathbf{S}_{V,E,s}$. Then Theorems 5.2 and 5.3 allow us to describe **SMod** *completely*, up to strict isomorphism of strict 2-categories, *using only differential geometric language:*

- Objects of **SMod** correspond to triples (V, E, s), with V a manifold, $E \to V$ a vector bundle, and $s \in C^\infty(E)$.
- 1-morphisms $(V, E, s) \to (W, F, t)$ correspond to equivalence classes $[V', f, \hat f]$ of triples $(V', f, \hat f)$, where V' is an open neighbourhood of $s^{-1}(0)$ in V, and $f : V' \to W$ is smooth, and $\hat f : E' \to f^*(F)$ is a morphism of vector bundles on V' with $\hat f \circ s' = f^*(t) + O(s^2)$, where $E' = E|_{V'}$, $s' = s|_{V'}$, and two triples $(V_1', f_1, \hat f_1), (V_2', f_2, \hat f_2)$ are equivalent if $f_2|_{V_1'\cap V_2'} = f_1|_{V_1'\cap V_2'} + O(s^2)$ and $\hat f_2|_{V_1'\cap V_2'} = \hat f_1|_{V_1'\cap V_2'} + O(s)$.

Conclusions

- 2-morphisms $[V_1', f_1, \hat{f}_1] \Rightarrow [V_2', f_2, \hat{f}_2]$ exist only if $f_2 = f_1 + O(s)$, and correspond to equivalence classes $[\Lambda]$ of vector bundle morphisms $\Lambda : E|_{V_1' \cap V_2'} \to f_1|^*_{V_1' \cap V_2'}(TW)$ with $f_2 = f_1 + \Lambda \cdot s + O(s^2)$ and $\hat{f}_2 = \hat{f}_1 + \Lambda \cdot f^*(\mathrm{d}t) + O(s)$, and Λ_1, Λ_2 are equivalent if $\Lambda_2 = \Lambda_1 + O(s)$.
- We can also give differential-geometric definitions of the other structures of a strict 2-category: composition of 1-morphisms, vertical and horizontal composition of 2-morphisms, identities. For example, the composition of 1-morphisms $[V', f, \hat{f}] : (V, E, s) \to (W, F, t)$ and $[W', g, \hat{g}] : (W, F, t) \to (X, G, u)$ is
$$[W', g, \hat{g}] \circ [V', f, \hat{f}] = \bigl[f^{-1}(W'), g \circ f|_{\dots}, f^{-1}(\hat{g}) \circ \hat{f}|_{\dots}\bigr],$$
and $\mathrm{id}_{(V,E,s)} = [V, \mathrm{id}_V, \mathrm{id}_E]$, and $\mathrm{id}_{[V',f,\hat{f}]} = [0]$.

Conclusions

So, if we are happy to work only in **SMod** $\subset$ **dMan**, that is, with d-manifolds which are covered by a single Kuranishi neighbourhood (V, E, s), we can give up all the tedious mucking about with (dg) C^∞-rings, sheaves, C^∞-schemes, etc., and work only with manifolds, vector bundles, and smooth sections. We do have to get used to the $O(s), O(s^2)$ notation, though.

Later in the course we will explain the following:

Theorem 5.4

Let $\mathbf{X}$ *be a d-manifold. Then* $\mathbf{X}$ *is equivalent in* **dMan** *to a standard model d-manifold* $\mathbf{S}_{V,E,s}$ *if and only if the dimensions of 'tangent spaces'* $\dim T_x\mathbf{X}$ *are globally bounded on* $\mathbf{X}$*. For instance, this is true if* $\mathbf{X}$ *is compact.*

Because of this, almost all interesting d-manifolds can be written in the form $\mathbf{S}_{V,E,s}$, and we lose little by working in **SMod**.

5.4. Tangent spaces and obstruction spaces

Let $\mathbf{X}$ be a d-manifold, and $x\in\mathbf{X}$. Then we have the *tangent space* $T_x\mathbf{X}$ and *obstruction space* $O_x\mathbf{X}$, which are natural finite-dimensional real vector spaces with $\dim T_x\mathbf{X}-\dim O_x\mathbf{X}=\mathrm{vdim}\,\mathbf{X}$. The dual vector spaces are the *cotangent space* $T_x^*\mathbf{X}$ and *coobstruction space* $O_x^*\mathbf{X}$. If $\mathbb{L}_{\mathbf{X}}=[\mathbb{L}_{\mathbf{X}}^{-1}\xrightarrow{\mathrm{d}}\mathbb{L}_{\mathbf{X}}^0]$ is the cotangent complex of $\mathbf{X}$ as a d-space, as in §4.3, we may define these by the exact sequence

$$0\longrightarrow O_x^*\mathbf{X}\longrightarrow \mathbb{L}_{\mathbf{X}}^{-1}|_x\xrightarrow{\mathrm{d}|_x}\mathbb{L}_{\mathbf{X}}^0|_x\longrightarrow T_x^*\mathbf{X}\longrightarrow 0. \qquad (5.2)$$

If $\mathbf{f}:\mathbf{X}\to\mathbf{Y}$ is a 1-morphism in **dMan** and $x\in\mathbf{X}$ with $\mathbf{f}(x)=y\in\mathbf{Y}$, we have natural, functorial linear maps $T_x\mathbf{f}:T_x\mathbf{X}\to T_y\mathbf{Y}$ and $O_x\mathbf{f}:O_x\mathbf{X}\to O_y\mathbf{Y}$. If $\eta:\mathbf{f}\Rightarrow\mathbf{g}$ is a 2-morphism in **dMan** then $T_x\mathbf{f}=T_x\mathbf{g}$ and $O_x\mathbf{f}=O_x\mathbf{g}$.

If $\mathbf{X}$ is a standard model d-manifold $\mathbf{S}_{V,E,s}$ then $\mathbb{L}_{\mathbf{X}}=[E^*|_{s^{-1}(0)}\xrightarrow{\mathrm{d}s}T^*V|_{s^{-1}(0)}]$. So dualizing (5.2), for each $x\in s^{-1}(0)\subseteq V$, the tangent and obstruction spaces are given by the exact sequence

$$0\longrightarrow T_x\mathbf{S}_{V,E,s}\longrightarrow T_xV\xrightarrow{\mathrm{d}s|_x}E|_x\longrightarrow O_x\mathbf{S}_{V,E,s}\longrightarrow 0. \qquad (5.3)$$

That is, $T_x\mathbf{S}_{V,E,s}$, $O_x\mathbf{S}_{V,E,s}$ are the kernel and cokernel of $\mathrm{d}s|_x:T_xV\to E|_x$. If $\mathbf{S}_{V',f,\hat f}:\mathbf{S}_{V,E,s}\to\mathbf{S}_{W,F,t}$ is a standard model 1-morphism then $T_x\mathbf{S}_{V',f,\hat f}$, $O_x\mathbf{S}_{V',f,\hat f}$ are given by the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & T_x\mathbf{S}_{V,E,s} \arrow[r] \arrow[d, "T_x\mathbf{S}_{V',f,\hat f}"] & T_xV \arrow[r, "\mathrm{d}s|_x"'] \arrow[d, "T_xf"] & E|_x \arrow[r] \arrow[d, "\hat f|_x"] & O_x\mathbf{S}_{V,E,s} \arrow[r] \arrow[d, "O_x\mathbf{S}_{V',f,\hat f}"] & 0 \\ 0 \arrow[r] & T_y\mathbf{S}_{W,F,t} \arrow[r] & T_yW \arrow[r, "\mathrm{d}t|_y"] & F|_y \arrow[r] & O_y\mathbf{S}_{W,F,t} \arrow[r] & 0. \end{tikzcd} \qquad (5.4)$$

Étale 1-morphisms and equivalences

Definition

A 1-morphism $\mathbf{f}:\mathbf{X}\to\mathbf{Y}$ in $\mathbf{dMan}$ or $\mathbf{dSpa}$ is called *étale* if it is a local equivalence. That is, $\mathbf{f}$ is étale if for all $x\in\mathbf{X}$ there exist open d-submanifolds $x\in\mathbf{U}\subseteq\mathbf{X}$ and $f(x)\in\mathbf{V}\subseteq\mathbf{Y}$ with $\mathbf{f}(\mathbf{U})=\mathbf{V}$, such that $\mathbf{f}|_{\mathbf{U}}:\mathbf{U}\to\mathbf{V}$ is an equivalence in the 2-category $\mathbf{dMan}$.

Theorem 5.5

A 1-morphism $\mathbf{f}:\mathbf{X}\to\mathbf{Y}$ in $\mathbf{dMan}$ or $\mathbf{dSpa}$ is an equivalence if and only if it is étale and $f:X\to Y$ is a bijection of sets.

The proof involves choosing local quasi-inverses $\mathbf{g}_i:\mathbf{V}_i\to\mathbf{U}_i$ for $\mathbf{f}|_{\mathbf{U}_i}:\mathbf{U}_i\to\mathbf{V}_i$ for $\{\mathbf{U}_i:i\in I\}$, $\{\mathbf{V}_i:i\in I\}$ open covers of $\mathbf{X},\mathbf{Y}$, and then gluing the $\mathbf{g}_i$ for $i\in I$ using a partition of unity to get a global quasi-inverse for $\mathbf{f}$.

Theorem 5.6

A 1-morphism $\mathbf{f}:\mathbf{X}\to\mathbf{Y}$ in $\mathbf{dMan}$ is étale if and only if $T_x\mathbf{f}:T_x\mathbf{X}\to T_y\mathbf{Y}$ and $O_x\mathbf{f}:O_x\mathbf{X}\to O_y\mathbf{Y}$ are isomorphisms for all $x\in\mathbf{X}$ with $\mathbf{f}(x)=y\in\mathbf{Y}$.

The 'only if' part is obvious: if $\mathbf{g}:\mathbf{V}\to\mathbf{U}$ is a local quasi-inverse for $\mathbf{f}$, then $T_y\mathbf{g}, O_y\mathbf{g}$ are inverses for $T_x\mathbf{f}, O_x\mathbf{f}$. For the 'if' part, replacing $\mathbf{X},\mathbf{Y},\mathbf{f}$ locally by 'standard model' d-manifolds and 1-morphism, we can construct an explicit quasi-inverse at the level of dg C^∞-rings by choosing a splitting of an exact sequence of vector bundles.

The analogue is false for $\mathbf{dSpa}$.

Combining (5.4) and Theorems 5.5 and 5.6 gives a criterion for when a standard model 1-morphism is étale or an equivalence:

Theorem 5.7

Let $\mathbf{S}_{V',f,\hat f}:\mathbf{S}_{V,E,s}\to\mathbf{S}_{W,F,t}$ be a standard model 1-morphism in **dMan**. *Then $\mathbf{S}_{V',f,\hat f}$ is étale if and only if for all $x\in s^{-1}(0)\subseteq V$ with $f(x)=y\in t^{-1}(0)\subseteq W$, the following sequence is exact:*

$$0 \longrightarrow T_xV \xrightarrow{\mathrm{d}s|_x\oplus T_xf} E|_x\oplus T_yW \xrightarrow{\hat f|_x\oplus -\mathrm{d}t|_y} F|_y \longrightarrow 0. \quad (5.5)$$

Also $\mathbf{S}_{V',f,\hat f}$ is an equivalence in **dMan** *if in addition $f|_{s^{-1}(0)}:s^{-1}(0)\to t^{-1}(0)$ is a bijection.*

Example 5.8

In Fukaya–Oh–Ohta–Ono Kuranishi spaces in symplectic geometry, a 'coordinate change' $(f,\hat f):(V,E,s)\to(W,F,t)$ of 'Kuranishi neighbourhoods' $(V,E,s),(W,F,t)$ is an embedding of submanifolds $f:V\hookrightarrow W$ and an embedding of vector bundles $\hat f:E\hookrightarrow f^*(F)$ with $\hat f\circ s=f^*(t)$, such that the induced morphism $(\mathrm{d}s)_*:f^*(TW)/TV\to f^*(F)/E$ is an isomorphism near $s^{-1}(0)$. Theorem 5.7 shows $\mathbf{S}_{V,f,\hat f}:\mathbf{S}_{V,E,s}\to\mathbf{S}_{W,F,t}$ is étale, or an equivalence. But FOOO coordinate changes are very special examples of equivalences; they only exist if $\dim V\leqslant\dim W$.

Derived Differential Geometry

Lecture 6 of 14: M-Kuranishi spaces

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Summer 2015

These slides, and references, etc., available at
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Plan of talk:

6. M-Kuranishi spaces

We now explain another way to define (2-)categories of derived manifolds, using an 'atlas of charts' approach, motivated by the ideas of §5. Today we will define an ordinary category **MKur** of 'M-Kuranishi spaces'. (The 'M-' stands for 'Manifold', following Hofer's 'M-polyfolds' and 'polyfolds'.)
Recall that *orbifolds* are generalizations of manifolds locally modelled on $\mathbb{R}^n/\Gamma$, for Γ a finite group acting linearly on $\mathbb{R}^n$. Later in the course we will define a weak 2-category **Kur** of 'Kuranishi spaces', a form of derived orbifold. The full 2-subcategory $\mathbf{Kur_{trG}} \subset \mathbf{Kur}$ of Kuranishi spaces with trivial orbifold groups is a 2-category of derived manifolds. There are equivalences of categories $\mathbf{MKur} \simeq \mathrm{Ho}(\mathbf{Kur_{trG}}) \simeq \mathrm{Ho}(\mathbf{dMan})$, where $\mathrm{Ho}(\mathbf{Kur_{trG}}), \mathrm{Ho}(\mathbf{dMan})$ are the homotopy categories, and an equivalence of weak 2-categories $\mathbf{Kur_{trG}} \simeq \mathbf{dMan}$.

In fact 'Kuranishi spaces' (with a different, non-equivalent definition, which we will call 'FOOO Kuranishi spaces') have been used for many years in the work of Fukaya et al. in symplectic geometry (Fukaya and Ono 1999, Fukaya–Oh–Ohta–Ono 2009), as the geometric structure on moduli spaces of J-holomorphic curves. There are problems with their theory (e.g. there is no notion of morphism of FOOO Kuranishi space), and I claim my definition is the 'correct' definition of Kuranishi space, which should replace the FOOO definition. Any FOOO Kuranishi space $\mathbf{X}$ can be made into a Kuranishi space $\mathbf{X}'$ in my sense, uniquely up to equivalence in the 2-category **Kur**. I began working in Derived Differential Geometry to try and find the 'correct' definition of Kuranishi space, and sort out the problems in the area.
To motivate the comparison between d-manifolds and M-Kuranishi spaces, consider the following two equivalent definitions of manifold:

Definition 6.1

A *manifold* of dimension n is a Hausdorff, second countable topological space X with a sheaf $\mathcal{O}_X$ of $\mathbb{R}$-algebras (or C^∞-rings) locally isomorphic to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$, where $\mathcal{O}_{\mathbb{R}^n}$ is the sheaf of smooth functions $f : \mathbb{R}^n \to \mathbb{R}$.

Definition 6.2

A *manifold* of dimension n is a Hausdorff, second countable topological space X equipped with an atlas of charts $\{(V_i, \psi_i) : i \in I\}$, where $V_i \subseteq \mathbb{R}^n$ is open, and $\psi_i : V_i \to X$ is a homeomorphism with an open subset $\mathrm{Im}\,\psi_i$ of X for all $i \in I$, and $\psi_j^{-1} \circ \psi_i : \psi_i^{-1}(\mathrm{Im}\,\psi_j) \to \psi_j^{-1}(\mathrm{Im}\,\psi_i)$ is a diffeomorphism of open subsets of $\mathbb{R}^n$ for all $i, j \in I$.

If you try to define derived manifolds by generalizing Definition 6.1, you get d-manifolds (or something similar, e.g. Spivak); if you try to generalize Definition 6.2, you get (M-)Kuranishi spaces.

6.1. M-Kuranishi neighbourhoods and their morphisms

Definition 6.3

Let X be a topological space. An *M-Kuranishi neighbourhood* on X is a quadruple (V, E, s, ψ) such that:

(a) V is a smooth manifold.
(b) $\pi : E \to V$ is a vector bundle over V, the *obstruction bundle*.
(c) $s \in C^\infty(E)$ is a smooth section of E, the *Kuranishi section*.
(d) ψ is a homeomorphism from $s^{-1}(0)$ to an open subset $\mathrm{Im}\,\psi$ in X, where $\mathrm{Im}\,\psi$ is called the *footprint* of (V, E, s, ψ).

If $S \subseteq X$ is open, we call (V, E, s, ψ) an *M-Kuranishi neighbourhood over S* if $S \subseteq \mathrm{Im}\,\psi \subseteq X$.

This is the same as Fukaya–Oh–Ohta–Ono Kuranishi neighbourhoods, omitting finite groups Γ.

Definition 6.4

Let X be a topological space, (V_i, E_i, s_i, ψ_i), (V_j, E_j, s_j, ψ_j) be M-Kuranishi neighbourhoods on X, and $S \subseteq \mathrm{Im}\,\psi_i \cap \mathrm{Im}\,\psi_j \subseteq X$ be an open set. Consider triples $(V_{ij}, \phi_{ij}, \hat\phi_{ij})$ satisfying:

(a) V_{ij} is an open neighbourhood of $\psi_i^{-1}(S)$ in V_i.

(b) $\phi_{ij} : V_{ij} \to V_j$ is smooth, with $\psi_i = \psi_j \circ \phi_{ij}$ on $s_i^{-1}(0) \cap V_{ij}$.

(c) $\hat\phi_{ij} : E_i|_{V_{ij}} \to \phi_{ij}^*(E_j)$ is a morphism of vector bundles on V_{ij}, with $\hat\phi_{ij}(s_i|_{V_{ij}}) = \phi_{ij}^*(s_j) + O(s_i^2)$.

Define an equivalence relation $\sim$ on such triples $(V_{ij}, \phi_{ij}, \hat\phi_{ij})$ by $(V_{ij}, \phi_{ij}, \hat\phi_{ij}) \sim (V'_{ij}, \phi'_{ij}, \hat\phi'_{ij})$ if there are open $\psi_i^{-1}(S) \subseteq \dot V_{ij} \subseteq V_{ij} \cap V'_{ij}$ and a morphism $\Lambda : E_i|_{\dot V_{ij}} \to \phi_{ij}^*(TV_j)|_{\dot V_{ij}}$ of vector bundles on $\dot V_{ij}$ satisfying $\phi'_{ij} = \phi_{ij} + \Lambda\cdot s_i + O(s_i^2)$ and $\hat\phi'_{ij} = \hat\phi_{ij} + \Lambda\cdot\phi_{ij}^*(\mathrm{d}s_j) + O(s_i)$ on $\dot V_{ij}$. We write $[V_{ij}, \phi_{ij}, \hat\phi_{ij}]$ for the $\sim$-equivalence class of $(V_{ij}, \phi_{ij}, \hat\phi_{ij})$, and call $[V_{ij}, \phi_{ij}, \hat\phi_{ij}] : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ a *morphism of M-Kuranishi neighbourhoods over* S.

We can interpret all this in terms of standard model d-manifolds from §5, and their 1- and 2-morphisms:

- An M-Kuranishi neighbourhood (V, E, s, ψ) on X corresponds to a standard model d-manifold $\mathbf{S}_{V,E,s}$ together with a homeomorphism ψ from the topological space $S_{V,E,s} = s^{-1}(0)$ to an open subset $\mathrm{Im}\,\psi \subseteq X$.
- A morphism $[V_{ij}, \phi_{ij}, \hat\phi_{ij}] : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ of M-Kuranishi neighbourhoods consists of an open d-submanifold $\mathbf{S}_{V_{ij},E_i|\ldots,s_i|\ldots} \subseteq \mathbf{S}_{V_i,E_i,s_i}$, together with a 2-isomorphism class $[\mathbf{S}_{V_{ij},\phi_{ij},\hat\phi_{ij}}]$ of standard model 1-morphisms $\mathbf{S}_{V_{ij},\phi_{ij},\hat\phi_{ij}} : \mathbf{S}_{V_{ij},E_i|\ldots,s_i|\ldots} \to \mathbf{S}_{V_j,E_j,s_j}$, such that on topological spaces we have $\psi_j \circ S_{V_{ij},\phi_{ij},\hat\phi_{ij}} = \psi_i : S_{V_{ij},E_i|\ldots,s_i|\ldots} \to X$.

  The definition of $(V_{ij}, \phi_{ij}, \hat\phi_{ij}) \sim (V'_{ij}, \phi'_{ij}, \hat\phi'_{ij})$ is just the existence of a 2-isomorphism $\mathbf{S}_\Lambda : \mathbf{S}_{V_{ij},\phi_{ij},\hat\phi_{ij}} \Rightarrow \mathbf{S}_{V'_{ij},\phi'_{ij},\hat\phi'_{ij}}$.

Given morphisms $[V_{ij},\phi_{ij},\hat\phi_{ij}]:(V_i,E_i,s_i,\psi_i)\to(V_j,E_j,s_j,\psi_j)$, $[V_{jk},\phi_{jk},\hat\phi_{jk}]:(V_j,E_j,s_j,\psi_j)\to(V_k,E_k,s_k,\psi_k)$ of M-Kuranishi neighbourhoods over $S\subseteq X$, the *composition* is

$$[V_{jk},\phi_{jk},\hat\phi_{jk}]\circ[V_{ij},\phi_{ij},\hat\phi_{ij}]=[\phi_{ij}^{-1}(V_{jk}),\phi_{jk}\circ\phi_{ij}|_{\cdots},\phi_{ij}^{-1}(\hat\phi_{jk})\circ\hat\phi_{ij}|_{\cdots}]:\\ (V_i,E_i,s_i,\psi_i)\longrightarrow(V_k,E_k,s_k,\psi_k).$$

Then M-Kuranishi neighbourhoods over $S\subseteq X$ form a category $\mathrm{MKur}_S(X)$. We call $[V_{ij},\phi_{ij},\hat\phi_{ij}]$ an *M-coordinate change over S* if it is an isomorphism in $\mathrm{MKur}_S(X)$. Theorem 5.7 implies:

Theorem 6.5

A morphism $[V_{ij},\phi_{ij},\hat\phi_{ij}]:(V_i,E_i,s_i,\psi_i)\to(V_j,E_j,s_j,\psi_j)$ is an M-coordinate change over S if and only if for all $x\in S$ with $v_i=\psi_i^{-1}(x)$ and $v_j=\psi_j^{-1}(x)$, the following sequence is exact:

$$0\longrightarrow T_{v_i}V_i\xrightarrow{\mathrm{d}s_i|_{v_i}\oplus T_{v_i}\phi_{ij}}E_i|_{v_i}\oplus T_{v_j}V_j\xrightarrow{\hat\phi_{ij}|_{v_i}\oplus -\mathrm{d}s_j|_{v_j}}E_j|_{v_j}\longrightarrow 0.$$

The sheaf property of morphisms

Theorem 6.6

Let (V_i,E_i,s_i,ψ_i), (V_j,E_j,s_j,ψ_j) be M-Kuranishi neighbourhoods on X. For each open $S\subseteq\mathrm{Im}\,\psi_i\cap\mathrm{Im}\,\psi_j$, write $\mathcal{H}om\bigl((V_i,E_i,s_i,\psi_i),(V_j,E_j,s_j,\psi_j)\bigr)(S)$ for the set of morphisms $\Phi_{ij}:(V_i,E_i,s_i,\psi_i)\to(V_j,E_j,s_j,\psi_j)$ over S, and for all open $T\subseteq S\subseteq\mathrm{Im}\,\psi_i\cap\mathrm{Im}\,\psi_j$ define

$$\rho_{ST}:\mathcal{H}om\bigl((V_i,E_i,s_i,\psi_i),(V_j,E_j,s_j,\psi_j)\bigr)(S)\longrightarrow\\ \mathcal{H}om\bigl((V_i,E_i,s_i,\psi_i),(V_j,E_j,s_j,\psi_j)\bigr)(T)\ \text{by}\ \rho_{ST}:\Phi_{ij}\longmapsto\Phi_{ij}|_T.$$

Then $\mathcal{H}om\bigl((V_i,E_i,s_i,\psi_i),(V_j,E_j,s_j,\psi_j)\bigr)$ is a sheaf of sets on $\mathrm{Im}\,\psi_i\cap\mathrm{Im}\,\psi_j$. Similarly, M-coordinate changes from (V_i,E_i,s_i,ψ_i) to (V_j,E_j,s_j,ψ_j) are a subsheaf of $\mathcal{H}om\bigl((V_i,E_i,s_i,\psi_i),(V_j,E_j,s_j,\psi_j)\bigr)$.

This is not obvious, but can be seen using the d-manifold interpretation. It means we can glue (iso)morphisms of M-Kuranishi neighbourhoods over the sets of an open cover.

We generalize Definition 6.4:

Definition 6.7

Let $f : X \to Y$ be a continuous map of topological spaces, (V_i, E_i, s_i, ψ_i), (W_j, F_j, t_j, χ_j) be M-Kuranishi neighbourhoods on X, Y, and $S \subseteq \mathrm{Im}\,\psi_i \cap f^{-1}(\mathrm{Im}\,\chi_j) \subseteq X$ be an open set. Consider triples $(V_{ij}, f_{ij}, \hat{f}_{ij})$ satisfying:

(a) V_{ij} is an open neighbourhood of $\psi_i^{-1}(S)$ in V_i.

(b) $f_{ij} : V_{ij} \to W_j$ is smooth, with $f \circ \psi_i = \chi_j \circ f_{ij}$ on $s_i^{-1}(0) \cap V_{ij}$.

(c) $\hat{f}_{ij} : E_i|_{V_{ij}} \to f_{ij}^*(F_j)$ is a morphism of vector bundles on V_{ij}, with $\hat{f}_{ij}(s_i|_{V_{ij}}) = f_{ij}^*(t_j) + O(s_i^2)$.

Define an equivalence relation $\sim$ by $(V_{ij}, f_{ij}, \hat{f}_{ij}) \sim (V'_{ij}, f'_{ij}, \hat{f}'_{ij})$ if there are open $\psi_i^{-1}(S) \subseteq \dot{V}_{ij} \subseteq V_{ij} \cap V'_{ij}$ and $\Lambda : E_i|_{\dot{V}_{ij}} \to f_{ij}^*(TW_j)|_{\dot{V}_{ij}}$ with $f'_{ij} = f_{ij} + \Lambda \cdot s_i + O(s_i^2)$ and $\hat{f}'_{ij} = \hat{f}_{ij} + \Lambda \cdot f_{ij}^*(\mathrm{d}t_j) + O(s_i)$. We write $[V_{ij}, f_{ij}, \hat{f}_{ij}]$ for the $\sim$-equivalence class of $(V_{ij}, f_{ij}, \hat{f}_{ij})$, and call $[V_{ij}, f_{ij}, \hat{f}_{ij}] : (V_i, E_i, s_i, \psi_i) \to (W_j, F_j, t_j, \chi_j)$ a *morphism over* S, f.

When $Y = X$ and $f = \mathrm{id}_X$, this recovers the notion of morphisms of M-Kuranishi neighbourhoods on X. We have the obvious notion of compositions of morphisms of M-Kuranishi neighbourhoods over $f : X \to Y$ and $g : Y \to Z$.

Here is the generalization of Theorem 6.6:

Theorem 6.8

Let (V_i, E_i, s_i, ψ_i), (W_j, F_j, t_j, χ_j) be M-Kuranishi neighbourhoods on X, Y, and $f : X \to Y$ be continuous. Then morphisms from (V_i, E_i, s_i, ψ_i) to (W_j, F_j, t_j, χ_j) over f form a sheaf $\mathcal{H}om_f\big((V_i, E_i, s_i, \psi_i), (W_j, F_j, t_j, \chi_j)\big)$ on $\mathrm{Im}\,\psi_i \cap f^{-1}(\mathrm{Im}\,\chi_j)$.

This will be essential for defining compositions of morphisms of M-Kuranishi spaces. The lack of such a sheaf property in the FOOO theory is why FOOO Kuranishi spaces are not a category.

6.2. M-Kuranishi spaces

Definition 6.9

Let X be a Hausdorff, second countable topological space, and $n \in \mathbb{Z}$. An *M-Kuranishi structure $\mathcal{K}$ on X of virtual dimension n* is data $\mathcal{K} = \big(I, (V_i, E_i, s_i, \psi_i)_{i\in I}, \Phi_{ij,\, i,j\in I}\big)$, where:

(a) I is an indexing set.

(b) (V_i, E_i, s_i, ψ_i) is an M-Kuranishi neighbourhood on X for each $i \in I$, with $\dim V_i - \operatorname{rank} E_i = n$.

(c) $\Phi_{ij} = [V_{ij}, \phi_{ij}, \hat{\phi}_{ij}] : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ is an M-coordinate change over $S = \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$ for all $i, j \in I$.

(d) $\bigcup_{i\in I} \operatorname{Im}\psi_i = X$.

(e) $\Phi_{ii} = \mathrm{id}_{(V_i,E_i,s_i,\psi_i)}$ for all $i \in I$.

(f) $\Phi_{jk} \circ \Phi_{ij} = \Phi_{ik}$ for all $i, j, k \in I$ over $S = \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j \cap \operatorname{Im}\psi_k$.

We call $\mathbf{X} = (X, \mathcal{K})$ an *M-Kuranishi space*, of *virtual dimension* $\operatorname{vdim}\mathbf{X} = n$. When we write $x \in \mathbf{X}$, we mean that $x \in X$.

In terms of standard model d-manifolds, an M-Kuranishi structure $\mathcal{K}$ on X is the data:

- An open cover $\{\operatorname{Im}\psi_i : i \in I\}$ of X.
- Standard model d-manifolds $\mathbf{S}_{V_i,E_i,s_i}$ for $i \in I$, with homeomorphisms $\psi_i : S_{V_i,E_i,s_i} \to \operatorname{Im}\psi_i \subseteq X$.
- On each double overlap $\operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$ for $i, j \in I$, a 2-isomorphism class $[\mathbf{S}_{V_{ij},\phi_{ij},\hat{\phi}_{ij}}]$ of equivalences $\mathbf{S}_{V_{ij},\phi_{ij},\hat{\phi}_{ij}} : \mathbf{S}_{V_{ij},E_i|\ldots,s_i|\ldots} \to \mathbf{S}_{V_{ji},E_j|\ldots,s_j|\ldots}$ in **dMan**, where $\mathbf{S}_{V_{ij},E_i|\ldots,s_i|\ldots} \subseteq \mathbf{S}_{V_i,E_i,s_i}$ and $\mathbf{S}_{V_{ji},E_j|\ldots,s_j|\ldots} \subseteq \mathbf{S}_{V_j,E_j,s_j}$ are the open d-submanifolds corresponding to $\operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$.
- On each triple overlap $\operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j \cap \operatorname{Im}\psi_k$, there must exist a 2-isomorphism $\mathbf{S}_{V_{jk},\phi_{jk},\hat{\phi}_{jk}} \circ \mathbf{S}_{V_{ij},\phi_{ij},\hat{\phi}_{ij}} \cong \mathbf{S}_{V_{ik},\phi_{ik},\hat{\phi}_{ik}}$.

In the 'atlas of charts' definition of manifolds, we provide data (V_i, ψ_i) on each set $\operatorname{Im}\psi_i$ of an open cover, and verify conditions on double overlaps $\operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$. Here we provide data on $\operatorname{Im}\psi_i$ and $\operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$, and verify conditions on $\operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j \cap \operatorname{Im}\psi_k$.

Definition 6.10

Let $\mathbf{X} = (X, \mathcal{K})$ with $\mathcal{K} = \big(I, (V_i, E_i, s_i, \psi_i)_{i\in I}, \Phi_{ii',\, i,i'\in I}\big)$ and $\mathbf{Y} = (Y, \mathcal{L})$ with $\mathcal{L} = \big(J, (W_j, F_j, t_j, \chi_j)_{j\in J}, \Psi_{jj',\, j,j'\in J}\big)$ be M-Kuranishi spaces. A *morphism* $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ is $\mathbf{f} = \big(f, \mathbf{f}_{ij,\, i\in I,\, j\in J}\big)$, where $f : X \to Y$ is a continuous map, and $\mathbf{f}_{ij} = [V_{ij}, f_{ij}, \hat f_{ij}] : (V_i, E_i, s_i, \psi_i) \to (W_j, F_j, t_j, \chi_j)$ is a morphism of M-Kuranishi neighbourhoods over $S = \mathrm{Im}\,\psi_i \cap f^{-1}(\mathrm{Im}\,\chi_j)$ and f for all $i \in I$, $j \in J$, satisfying the conditions:

(a) If $i, i' \in I$ and $j \in J$ then $\mathbf{f}_{i'j} \circ \Phi_{ii'}|_S = \mathbf{f}_{ij}|_S$ over $S = \mathrm{Im}\,\psi_i \cap \mathrm{Im}\,\psi_{i'} \cap f^{-1}(\mathrm{Im}\,\chi_j)$ and f.

(b) If $i \in I$ and $j, j' \in J$ then $\Psi_{jj'} \circ \mathbf{f}_{ij}|_S = \mathbf{f}_{ij'}|_S$ over $S = \mathrm{Im}\,\psi_i \cap f^{-1}(\mathrm{Im}\,\chi_j \cap \mathrm{Im}\,\chi_{j'})$ and f.

If $x \in \mathbf{X}$ (i.e. $x \in X$), we will write $\mathbf{f}(x) = f(x) \in \mathbf{Y}$. When $\mathbf{Y} = \mathbf{X}$, so that $J = I$, define the identity morphism $\mathbf{id}_{\mathbf{X}} : \mathbf{X} \to \mathbf{X}$ by $\mathbf{id}_{\mathbf{X}} = \big(\mathrm{id}_X, \Phi_{ij,\, i,j\in I}\big)$.

Composition of morphisms

Let $\mathbf{X} = (X, \mathcal{I})$ with $\mathcal{I} = \big(I, (U_i, D_i, r_i, \phi_i)_{i\in I}, \Phi_{ii',\, i,i'\in I}\big)$ and $\mathbf{Y} = (Y, \mathcal{J})$ with $\mathcal{J} = \big(J, (V_j, E_j, s_j, \psi_j)_{j\in J}, \Psi_{jj',\, j,j'\in J}\big)$ and $\mathbf{Z} = (Z, \mathcal{K})$ with $\mathcal{K} = \big(K, (W_k, F_k, t_k, \xi_k)_{k\in K}, \Xi_{kk',\, k,k'\in K}\big)$ be M-Kuranishi spaces, and $\mathbf{f} = (f, \mathbf{f}_{ij}) : \mathbf{X} \to \mathbf{Y}$, $\mathbf{g} = (g, \mathbf{g}_{jk}) : \mathbf{Y} \to \mathbf{Z}$ be morphisms. Consider the problem of how to define the composition $\mathbf{g} \circ \mathbf{f} : \mathbf{X} \to \mathbf{Y}$.

For all $i \in I$ and $k \in K$, $\mathbf{g} \circ \mathbf{f}$ must contain a morphism $(\mathbf{g} \circ \mathbf{f})_{ik} : (U_i, D_i, r_i, \phi_i) \to (W_k, F_k, t_k, \xi_k)$ defined over $S_{ik} = \mathrm{Im}\,\phi_i \cap (g \circ f)^{-1}(\mathrm{Im}\,\xi_k)$ and $g \circ f$.

For each $j \in J$, we have a morphism $\mathbf{g}_{jk} \circ \mathbf{f}_{ij} : (U_i, D_i, r_i, \phi_i) \to (W_k, F_k, t_k, \xi_k)$, but it is defined over $S_{ijk} = \mathrm{Im}\,\phi_i \cap f^{-1}(\mathrm{Im}\,\psi_j) \cap (g \circ f)^{-1}(\mathrm{Im}\,\xi_k)$ and $g \circ f$, not over the whole of $S_{ik} = \mathrm{Im}\,\phi_i \cap (g \circ f)^{-1}(\mathrm{Im}\,\xi_k)$.

Composition of morphisms

The solution is to use the sheaf property of morphisms, Theorem 6.8. The sets S_{ijk} for $j \in J$ form an open cover of S_{ik}. Using Definition 6.10(a),(b) we can show that $\mathbf{g}_{jk} \circ \mathbf{f}_{ij}|_{S_{ijk} \cap S_{ij'k}} = \mathbf{g}_{j'k} \circ \mathbf{f}_{ij'}|_{S_{ijk} \cap S_{ij'k}}$. Therefore by Theorem 6.8 there is a unique morphism of M-Kuranishi neighbourhoods $(\mathbf{g} \circ \mathbf{f})_{ik} : (U_i, D_i, r_i, \phi_i) \to (W_k, F_k, t_k, \xi_k)$ defined over S_{ik} and $g \circ f$ with $(\mathbf{g} \circ \mathbf{f})_{ik}|_{S_{ijk}} = \mathbf{g}_{jk} \circ \mathbf{f}_{ij}$ for all $j \in J$. We show that $\mathbf{g} \circ \mathbf{f} := \big(g \circ f, (\mathbf{g} \circ \mathbf{f})_{ik,\ i \in I,\ k \in K}\big)$ is a morphism $\mathbf{g} \circ \mathbf{f} : \mathbf{X} \to \mathbf{Z}$ of M-Kuranishi spaces, which we call *composition*.

Composition is associative, and makes M-Kuranishi spaces into an ordinary category **MKur**.

Using facts about standard model d-manifolds, we can prove that there is an equivalence of categories $\mathbf{MKur} \simeq \mathrm{Ho}(\mathbf{dMan})$. Thus, isomorphism classes of M-Kuranishi spaces are in 1-1 correspondence with equivalence classes of d-manifolds.

Why no higher categories?

I have been stressing throughout that to do derived geometry properly, you should work in a higher category (a 2-category, or an ∞-category) rather than an ordinary category. So why have I just defined an ordinary category **MKur** of derived manifolds?

One answer is that you can always reduce to ordinary categories by taking homotopy categories, just as $\mathbf{MKur} \simeq \mathrm{Ho}(\mathbf{dMan})$. But doing so loses important information that we want to keep, and this information is missing in **MKur**. For example, fibre products $\mathbf{X} \times_{\mathbf{g},\mathbf{Z},\mathbf{h}} \mathbf{Y}$ in **MKur**, if they exist, will generally not be the 'correct' fibre products we want for applications, because the 'correct' fibre products are characterized by a universal property involving 2-morphisms, that makes no sense in **MKur**.

I only intended **MKur** as a 'cheap' version of derived manifolds, in which we sacrifice some good behaviour for the sake of simplicity.

Why no higher categories?

However, there is more to it than this. It is surprising that our definition of **MKur** 'works' at all, in the sense that it satisfies $\mathbf{MKur} \simeq \mathrm{Ho}(\mathbf{dMan}) \simeq \mathrm{Ho}(\mathbf{DerMan_{Spi}})$, so it is equivalent to the homotopy categories of some genuine higher categories of derived manifolds $\mathbf{dMan}, \mathbf{DerMan_{Spi}}$.

The reason for this is the complicated result Theorem 4.6 in §4 on gluing families of d-spaces $\mathbf{X}_i$, $i \in I$ (and hence d-manifolds) by equivalences on overlaps. Surprisingly, this theorem held in the homotopy category $\mathrm{Ho}(\mathbf{dSpa}), \mathrm{Ho}(\mathbf{dMan})$. That is, though we need the 2-category structure on **dMan** to form 'correct' fibre products, etc., we only need the ordinary category $\mathrm{Ho}(\mathbf{dMan})$ to glue by equivalences. The analogue is false for stacks, orbifolds, derived schemes, ... An M-Kuranishi space is basically a family of standard model d-manifolds $\mathbf{S}_{V_i,E_i,s_i}$ glued by equivalences on overlaps, in the homotopy category $\mathrm{Ho}(\mathbf{dMan})$.

6.3. Geometry of M-Kuranishi spaces

Example 6.11

Let X be a manifold. Then $(V, E, s, \psi) = (X, 0, 0, \mathrm{id}_X)$ is an M-Kuranishi neighbourhood on X, where $V = X$, $E = 0$ is the zero vector bundle on X, $s = 0$ is the zero section, and $\psi = \mathrm{id}_X : s^{-1}(0) = X \to X$. Define an M-Kuranishi structure $\mathcal{K} = (\{0\}, (X, 0, 0, \mathrm{id}_X)_0, \mathrm{id}_{(X,0,0,\mathrm{id}_X)\,00})$ on X to have indexing set $I = \{0\}$, one M-Kuranishi neighbourhood $(V_0, E_0, s_0, \psi_0) = (X, 0, 0, \mathrm{id}_X)$, and one M-coordinate change $\Phi_{00} = \mathrm{id}_{(X,0,0,\mathrm{id}_X)}$. Then $\mathbf{X} = (X, \mathcal{K})$ is an M-Kuranishi space.

Similarly, any smooth map of manifolds $f : X \to Y$ induces a morphism of M-Kuranishi spaces $\mathbf{f} = (f, \mathbf{f}_{00}) : \mathbf{X} \to \mathbf{Y}$ with $\mathbf{f}_{00} = [X, f, 0]$. This defines a full and faithful functor $F_{\mathbf{Man}}^{\mathbf{MKur}} : \mathbf{Man} \to \mathbf{MKur}$ mapping $X \mapsto \mathbf{X}$, $f \mapsto \mathbf{f}$, which embeds **Man** as a full subcategory of **MKur**. We say that an M-Kuranishi space $\mathbf{X}$ *is a manifold* if $\mathbf{X} \cong F_{\mathbf{Man}}^{\mathbf{MKur}}(X')$ for some manifold X'.

As for d-manifolds, for an M-Kuranishi space $\mathbf{X}$ we can define the *tangent space* $T_x\mathbf{X}$ and *obstruction space* $O_x\mathbf{X}$ for any $x \in \mathbf{X}$, where if $\mathbf{X} = (X, \mathcal{K})$ with $\mathcal{K} = \bigl(I, (V_i, E_i, s_i, \psi_i)_{i\in I}, \Phi_{ii',\ i,i'\in I}\bigr)$ and $x \in \operatorname{Im}\psi_i$ with $\psi_i^{-1}(x) = v_i \in s_i^{-1}(0) \subseteq V_i$ then as for (5.3) we have an exact sequence

$$0 \longrightarrow T_x\mathbf{X} \longrightarrow T_{v_i}V_i \xrightarrow{\mathrm{d}s_i|_{v_i}} E_i|_{v_i} \longrightarrow O_x\mathbf{X} \longrightarrow 0. \tag{6.1}$$

If $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ is a morphism of M-Kuranishi spaces we get functorial linear maps $T_x\mathbf{f} : T_x\mathbf{X} \to T_y\mathbf{Y}$ and $O_x\mathbf{f} : O_x\mathbf{X} \to O_y\mathbf{Y}$.

Theorem 6.12

(a) *An M-Kuranishi space* $\mathbf{X}$ *is a manifold iff* $O_x\mathbf{X}=0$ *for all* $x\in\mathbf{X}$.
(b) *A morphism* $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ *of M-Kuranishi spaces is étale (a local isomorphism) iff* $T_x\mathbf{f} : T_x\mathbf{X} \to T_y\mathbf{Y}$ *and* $O_x\mathbf{f} : O_x\mathbf{X} \to O_y\mathbf{Y}$ *are isomorphisms for all* $x \in \mathbf{X}$ *with* $\mathbf{f}(x) = y \in \mathbf{Y}$*. And* $\mathbf{f}$ *is an isomorphism in* **MKur** *if also* $f : X \to Y$ *is a bijection.*

Derived Differential Geometry

Lecture 7 of 14: Orbifolds

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Summer 2015

These slides available at
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Plan of talk:

7. Orbifolds

7.1. Introduction

Orbifolds $\mathfrak{X}$ are generalizations of manifolds which are locally modelled on $\mathbb{R}^n/\Gamma$ for Γ a finite group acting linearly on $\mathbb{R}^n$.

If Γ acts *effectively* on $\mathbb{R}^n$ (i.e. the morphism $\Gamma \to \mathrm{GL}(n,\mathbb{R})$ is injective, so that Γ is a subgroup of $\mathrm{GL}(n,\mathbb{R})$) then $\mathfrak{X}$ is called an *effective orbifold*. Some authors include this in the definition.

Orbifolds were introduced in 1956 by Satake, who called them 'V-manifolds'. Thurston gave them the name 'orbifolds' in 1980.

Lots of differential geometry for manifolds also works for orbifolds, often with only minor changes. Orbifolds are important in some kinds of 'moduli space' and 'invariant' theories, particularly for J-holomorphic curves in symplectic geometry and Gromov–Witten invariants, where one must "count" 'Deligne–Mumford stable curves' with finite symetry groups Γ, which makes the moduli spaces (derived) orbifolds rather than (derived) manifolds.

There are some subtle issues around defining 'smooth maps' of orbifolds, and so making orbifolds into a category (or higher category), and there are several *non-equivalent* definitions in the literature, both 'good' and 'bad'. For the 'bad' definitions, some differential-geometric operations such as transverse fibre products, or pullbacks of vector bundles, are not always defined.

The best answer is that orbifolds form a 2-*category* **Orb**, in which all 2-morphisms are 2-isomorphisms (i.e. a (2,1)-category).

Orbifolds are a kind of differential-geometric stack, and stacks form (2,1)-categories. There are at least five definitions of (strict or weak) 2-categories of orbifolds, giving equivalent 2-categories.

'Good' definitions of ordinary categories of orbifolds yield a category equivalent to $\mathrm{Ho}(\mathbf{Orb})$. In $\mathrm{Ho}(\mathbf{Orb})$, morphisms $[\mathfrak{f}] : \mathfrak{X} \to \mathfrak{Y}$ are not local (do not form a sheaf) on $\mathfrak{X}$. If you try to define an ordinary category of orbifolds in which smooth maps $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ are local on $\mathfrak{X}$, you get a 'bad' definition.

To see what these issues are, suppose $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ is a smooth map of orbifolds, and $x \in \mathfrak{X}$, $y \in \mathfrak{Y}$ with $\mathfrak{f}(x) = y$, and $\mathfrak{X}, \mathfrak{Y}$ are modelled near x, y on $[U/\Gamma]$, $[V/\Delta]$ for $U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^n$ open and Γ, Δ finite groups acting linearly on $\mathbb{R}^m, \mathbb{R}^n$ preserving U, V.
Naïvely, we would expect $\mathfrak{f}$ to be locally given near x by a smooth map of manifolds $f' : U \to V$ and a group morphism $\rho : \Gamma \to \Delta$ such that $f'(\gamma \cdot u) = \rho(\gamma) \cdot f'(u)$ for all $u \in U$ and $\gamma \in \Gamma$, so that f' induces a map of sets $U/\Gamma \to V/\Delta$.
Note that the map of sets $\mathfrak{f} : U/\Gamma \to V/\Delta$ does not determine f' and ρ uniquely. For any $\delta \in \Delta$, we can always replace f', ρ by $\tilde{f}', \tilde{\rho}$ where $\tilde{f}'(u) = \delta \cdot f'(u)$ and $\tilde{\rho}(\gamma) = \delta\rho(\gamma)\delta^{-1}$. For some $\mathfrak{f}$, there is more choice of f', ρ than this.
The definition of smooth map $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ needs to remember some information about allowed choices of (f', ρ). To see this is *2-categorical*, think of $(f', \rho), (\tilde{f}', \tilde{\rho})$ as 1-morphisms $(U, \Gamma) \to (V, \Delta)$, and $\delta : (f', \rho) \Rightarrow (\tilde{f}', \tilde{\rho})$ as a 2-morphism.

We will discuss some examples before formally defining orbifolds.

Example 7.1

Let X be a manifold, and G a finite group, so that $[*/G]$ is a (noneffective) orbifold. What are 'smooth maps' $\mathfrak{f} : X \to [*/G]$?
The answer should be: in the 2-category **Orb**,

- 1-morphisms $\mathfrak{f} : X \to [*/G]$ should correspond to principal G-bundles $P \to X$.
- For 1-morphisms $\mathfrak{f}, \tilde{\mathfrak{f}} : X \to [*/G]$ corresponding to principal G-bundles $P, \tilde{P} \to X$, 2-morphisms $\eta : \mathfrak{f} \Rightarrow \tilde{\mathfrak{f}}$ should correspond to isomorphisms of principal bundles $P \cong \tilde{P}$.

Therefore in the homotopy category $\mathrm{Ho}(\textbf{Orb})$, morphisms $[\mathfrak{f}] : X \to [*/G]$ correspond to isomorphism classes of principal G-bundles $P \to X$.

Non-locality of morphisms in $\mathrm{Ho}(\mathbf{Orb})$

Example 7.2

Let $X=\mathcal{S}^1\subset\mathbb{R}^2$ and $G=\mathbb{Z}_2$. Then $X=U\cup V$ for $U=\mathcal{S}^1\setminus\{(1,0)\}$, $V=\mathcal{S}^1\setminus\{(-1,0)\}$. There are two principal $\mathbb{Z}_2$-bundles on $\mathcal{S}^1$ up to isomorphism, with monodromy 1 and -1 around $\mathcal{S}^1$. But on $U\cong\mathbb{R}\cong V$ there are only one principal $\mathbb{Z}_2$-bundle (the trivial bundle) up to isomorphism.
Therefore morphisms $[\mathfrak{f}]:\mathcal{S}^1\to[*/\mathbb{Z}_2]$ in $\mathrm{Ho}(\mathbf{Orb})$ are not determined by their restrictions $[\mathfrak{f}]|_U,[\mathfrak{f}]|_V$ for the open cover $\{U,V\}$ of $\mathcal{S}^1$, so such $[\mathfrak{f}]$ do not form a sheaf on $\mathcal{S}^1$.

Regarding X as a quotient $[\mathcal{S}^1/\{1\}]$, this example also shows that morphisms $\mathfrak{f}:[U/\Gamma]\to[V/\Delta]$ in **Orb** or $\mathrm{Ho}(\mathbf{Orb})$ are *not* globally determined by a smooth map $f':U\to V$ and morphism $\rho:\Gamma\to\Delta$, as $f':\mathcal{S}^1\to *$, $\rho:\{1\}\to\mathbb{Z}_2$ are unique, but $\mathfrak{f}$ is not.

Example 7.3 (Hilsum–Skandalis morphisms)

Suppose U,V are manifolds and Γ,Δ are finite groups acting smoothly on U,V, so that $\mathfrak{X}=[U/\Gamma]$, $\mathfrak{Y}=[V/\Delta]$ are orbifolds ('global quotient orbifolds'). The correct notion of 1-morphism $\mathfrak{X}\to\mathfrak{Y}$ in **Orb** is induced by a triple (P,π,f), where

- P is a manifold with a smooth action of $\Gamma\times\Delta$
- $\pi:P\to U$ is a Γ-equivariant, Δ-invariant smooth map making P into a principal Δ-bundle over U.
- $f:P\to V$ is a smooth Δ-equivariant and Γ-invariant map.

This is called a *Hilsum–Skandalis morphism*.
2-morphisms $\eta:(P,\pi,f)\Rightarrow(\tilde P,\tilde\pi,\tilde f)$ are $\Gamma\times\Delta$-equivariant diffeomorphisms $\eta:P\to\tilde P$ with $\tilde\pi\circ\eta=\pi$, $\tilde f\circ\eta=f$.
If $(Q,\pi,g):[V/\Delta]\to[W/K]$ is another morphism then composition of 1-morphisms is

$$(Q,\pi,g)\circ(P,\pi,f)=\big((P\times_{f,V,\pi}Q)/\Delta,\pi\circ\pi_P,g\circ\pi_Q\big),$$

where $P\times_{f,V,\pi}Q$ is a transverse fibre product of manifolds.

Suppose $(P, \pi, f) : [U/\Gamma] \to [V/\Delta]$ is a Hilsum–Skandalis morphism with U connected, and $P = U \times \Delta$ is a trivial Δ-bundle, with Δ-action $\delta : (u, \delta') \mapsto (u, \delta\delta')$. The Γ-action on P commutes with the Δ-action and $\pi_U : U \times \Delta \to U$ is Γ-equivariant, so it must be of the form $\gamma : (u, \delta) \mapsto (\gamma \cdot u, \delta\rho(\gamma)^{-1})$ for $\rho : \Gamma \to \Delta$ a group morphism. Define $f' : U \to V$ by $f'(u) = f(u, 1)$. Then f Δ-equivariant implies that $f(u, \delta) = \delta \cdot f'(u)$, and f Γ-invariant implies that $f'(\gamma \cdot u) = \rho(\gamma) \cdot f'(u)$.
Thus, if P is a trivial Δ-bundle then (P, π, f) corresponds to the 'naïve' notion of morphisms $[U/\Gamma] \to [V/\Delta]$ discussed before.
Since every principal Δ-bundle is locally trivial, every Hilsum–Skandalis morphism is locally of the expected 'naïve' form.

Example 7.4 (Weighted projective spaces)

Let n and $a_0, \ldots, a_n$ be positive integers, with $\mathrm{hcf}(a_0, \ldots, a_n) = 1$. Define the *weighted projective space* $\mathbb{CP}^n_{a_0,\ldots,a_n} = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$, where $\mathbb{C}^*$ acts on $\mathbb{C}^{n+1} \setminus \{0\}$ by

$$\lambda : (z_0, z_1, \ldots, z_n) \longmapsto (\lambda^{a_0} z_0, \ldots, \lambda^{a_n} z_n).$$

Then $\mathbb{CP}^n_{a_0,\ldots,a_n}$ is a compact complex orbifold. Near $[z_0, \ldots, z_n]$ it is modelled on $\mathbb{C}^n/\mathbb{Z}_k$, where k is the highest common factor of those a_i for $i = 0, \ldots, n$ with $z_i \neq 0$.

Example 7.5

$\mathbb{CP}^2_{2,1}$ is topologically a 2-sphere $\mathcal{S}^2$. It has one orbifold point $[1,0]$ where it is locally modelled on $\mathbb{C}/\{\pm 1\}$, and $\mathbb{CP}^2_{2,1}\setminus[1,0]\cong\mathbb{C}$.
Suppose for a contradiction that $\mathbb{CP}^2_{2,1}\cong[U/\Gamma]$ for U a manifold and Γ a finite group. Let $[1,0]\cong u\Gamma$, and let $\Delta\subseteq\Gamma$ be the subgroup fixing u. Then Δ acts freely on U with $\Gamma/\Delta\cong\mathbb{Z}_2$. Let $U'=U\setminus u\Gamma$. Then $U'/\Delta\rightarrow U'/\Gamma\cong\mathbb{C}$ is a principal $\mathbb{Z}_2$-bundle, which must be trivial as $\mathbb{C}$ is simply-connected. But near $u\Gamma$, it should be modelled on $\mathbb{C}\setminus\{0\}\rightarrow(\mathbb{C}\setminus\{0\})/\mathbb{Z}_2$, which is a nontrivial principal $\mathbb{Z}_2$-bundle, a contradiction.
Thus $\mathbb{CP}^2_{2,1}$ cannot be a global quotient $[U/\Gamma]$ for Γ finite.

This example shows we need to define orbifolds $\mathfrak{X}$ by covering $\mathfrak{X}$ by many open charts $\mathfrak{U}_i\subset\mathfrak{X}$ with $\mathfrak{U}_i\cong[U_i/\Gamma_i]$.

7.2. Orbifold charts and coordinate changes

We now give one definition of a weak 2-category of orbifolds **Orb**, taken from my arXiv:1409.6908, §4.5. It is a dry run for the definition of Kuranishi spaces **Kur**.

Definition 7.6

Let X be a topological space. An *orbifold chart* (V_i,Γ_i,ψ_i) on X is a manifold V_i, a finite group Γ_i acting smoothly on V_i, and a map $\psi_i:V_i/\Gamma_i\rightarrow X$ which is a homeomorphism with an open set $\mathrm{Im}\,\psi_i\subseteq X$. We write $\bar\psi_i:V_i\rightarrow X$ for the composition $V_i\rightarrow V_i/\Gamma_i\xrightarrow{\psi_i}X$. If $X'\subseteq X$ is open, the *restriction* of (V_i,Γ_i,ψ_i) to X' is $(V_i,\Gamma_i,\psi_i)|_{X'}:=(V_i',\Gamma_i,\psi_i')$, where $V_i'=\bar\psi_i^{-1}(X')$ and $\psi_i'=\psi_i|_{V_i'/\Gamma_i}$.

Definition 7.7

Let $(V_i,\Gamma_i,\psi_i),(V_j,\Gamma_j,\psi_j)$ be orbifold charts on X with $\mathrm{Im}\,\psi_i=\mathrm{Im}\,\psi_j$. A *coordinate change* $(P_{ij},\pi_{ij},\phi_{ij}):(V_i,\Gamma_i,\psi_i)\to(V_j,\Gamma_j,\psi_j)$ is $P_{ij},\pi_{ij},\phi_{ij}$, where

- P_{ij} is a manifold with a smooth action of $\Gamma_i\times\Gamma_j$.
- $\pi_{ij}:P\to V_i$ is a Γ_i-equivariant, Γ_j-invariant smooth map making P_{ij} into a principal Γ_j-bundle over V_i.
- $\phi_{ij}:P_{ij}\to V_j$ is a Γ_j-equivariant, Γ_i-invariant smooth map making P_{ij} into a principal Γ_i-bundle over V_j.

If $(P_{ij},\pi_{ij},\phi_{ij}),(\tilde P_{ij},\tilde\pi_{ij},\tilde\phi_{ij}):(V_i,\Gamma_i,\psi_i)\to(V_j,\Gamma_j,\psi_j)$ are coordinate changes, a *2-morphism* $\eta:(P_{ij},\pi_{ij},\phi_{ij})\Rightarrow(\tilde P_{ij},\tilde\pi_{ij},\tilde\phi_{ij})$ is a $\Gamma_i\times\Gamma_j$-equivariant diffeomorphism $\eta:P_{ij}\to\tilde P_{ij}$ with $\tilde\pi_{ij}\circ\eta=\pi_{ij}$, $\tilde\phi_{ij}\circ\eta=\phi_{ij}$. If $X'\subseteq X$ is open, we can *restrict* coordinate changes and 2-morphisms to X' by $(P_{ij},\pi_{ij},\phi_{ij})|_{X'}:=((\bar\psi_i\circ\pi_{ij})^{-1}(X'),\pi_{ij}|_{\cdots},\phi_{ij}|_{\cdots})$ and $\eta|_{X'}:=\eta|_{(\bar\psi_i\circ\pi_{ij})^{-1}(X')}$.

Definition 7.8

If $(P_{jk},\pi_{jk},\phi_{jk}):(V_j,\Gamma_j,\psi_j)\to(V_k,\Gamma_k,\psi_k)$ is another coordinate change then *composition of coordinate changes* is
$$(P_{jk},\pi_{jk},\phi_{jk})\circ(P_{ij},\pi_{ij},\phi_{ij})=((P_{ij}\times_{\phi_{ij},V_j,\pi_{jk}}P_{jk})/\Gamma_j,\pi_{ij}\circ\pi_{P_{ij}},\phi_{jk}\circ\pi_{P_{jk}}),$$
where $P_{ij}\times_{\phi_{ij},V_j,\pi_{jk}}P_{jk}$ is a transverse fibre product of manifolds.

If $\eta:(P_{ij},\pi_{ij},\phi_{ij})\Rightarrow(\tilde P_{ij},\tilde\pi_{ij},\tilde\phi_{ij})$ and $\zeta:(\tilde P_{ij},\tilde\pi_{ij},\tilde\phi_{ij})\to(\hat P_{ij},\hat\pi_{ij},\hat\phi_{ij})$ are 2-morphisms, the *vertical composition* is $\zeta\odot\eta=\zeta\circ\eta:P_{ij}\to\hat P_{ij}$.

If $\eta:(P_{ij},\pi_{ij},\phi_{ij})\Rightarrow(\tilde P_{ij},\tilde\pi_{ij},\tilde\phi_{ij})$ and $\zeta:(P_{jk},\pi_{jk},\phi_{jk})\Rightarrow(\tilde P_{jk},\tilde\pi_{jk},\tilde\phi_{jk})$ are 2-morphisms, the *horizontal composition* is
$$\zeta*\eta=(\eta\times_{V_j}\zeta)/\Gamma_j:(P_{ij}\times_{V_j}P_{jk})/\Gamma_j\to(\tilde P_{ij}\times_{V_j}\tilde P_{jk})/\Gamma_j.$$

The *identity coordinate change* for (V_i,Γ_i,ψ_i) is $\mathrm{id}_{(V_i,\Gamma_i,\psi_i)}:=(P_{ii},\pi_{ii},\phi_{ii})$ where $P_{ii}=V_i\times\Gamma_i$ with $\Gamma_i\times\Gamma_i$-action $(\gamma_1,\gamma_2):(v,\delta)\mapsto(\gamma_1\cdot v,\gamma_2\delta\gamma_1^{-1})$, and $\pi_{ii}:(v,\gamma)\mapsto v$, $\phi_{ii}:(v,\gamma)\mapsto\gamma\cdot v$.

The *identity 2-morphism* for $(P_{ij},\pi_{ij},\phi_{ij})$ is $\mathrm{id}_{(P_{ij},\pi_{ij},\phi_{ij})}=\mathrm{id}_{P_{ij}}$.

Theorem 7.9

Let X be a topological space, and $S \subseteq X$ be open. Then we have defined a strict 2-category $\mathbf{Coord}_S(X)$ *with objects orbifold charts* (V_i, Γ_i, ψ_i) *on X with* $\operatorname{Im}\psi_i = S$, *and 1-morphisms coordinate changes* $(P_{ij}, \pi_{ij}, \phi_{ij}) : (V_i, \Gamma_i, \psi_i) \to (V_j, \Gamma_j, \psi_j)$, *and 2-morphisms* $\eta : (P_{ij}, \pi_{ij}, \phi_{ij}) \Rightarrow (\tilde P_{ij}, \tilde\pi_{ij}, \tilde\phi_{ij})$ *as above. All 1-morphisms in* $\mathbf{Coord}_S(X)$ *are 1-isomorphisms, and all 2-morphisms are 2-isomorphisms. If $T \subseteq S \subseteq X$ are open, then restriction* $|_T : \mathbf{Coord}_S(X) \to \mathbf{Coord}_T(X)$ *is a strict 2-functor.*

7.3. Stacks on topological spaces

In §3.2 we defined *sheaves of sets* $\mathcal{E}$ on a topological space X. There is a parallel notion of 'sheaves of groupoids' on X, which is called a *stack* (or 2-*sheaf*) on X. As sets form a category **Sets**, but groupoids form a 2-category **Groupoids** (in fact, a (2,1)-category), stacks on X are a (2,1)-category generalization of sheaves.

The connection with stacks in algebraic geometry is that both are examples of 'stacks on a site', where here we mean the site of open sets in X, and in algebraic geometry we use the site of $\mathbb{K}$-algebras $\mathbf{Alg}_{\mathbb{K}}$, regarded as a kind of generalized topological space.

As for sheaves, we define *prestacks* and *stacks*. Sheaves are presheaves which satisfy a gluing property on open covers $\{V_i : i \in I\}$, involving data on V_i and conditions on double overlaps $V_i \cap V_j$. For the 2-category generalization we need data on V_i, $V_i \cap V_j$ and conditions on triple overlaps $V_i \cap V_j \cap V_k$.

Definition 7.10

Let X be a topological space. A *prestack* (or *prestack in groupoids*, or 2-*presheaf*) $\mathcal{E}$ on X, consists of the data of a groupoid $\mathcal{E}(S)$ for every open set $S \subseteq X$, and a functor $\rho_{ST} : \mathcal{E}(S) \to \mathcal{E}(T)$ called the *restriction map* for every inclusion $T \subseteq S \subseteq X$ of open sets, and a natural isomorphism of functors $\eta_{STU} : \rho_{TU} \circ \rho_{ST} \Rightarrow \rho_{SU}$ for all inclusions $U \subseteq T \subseteq S \subseteq X$ of open sets, satisfying the conditions that:

(i) $\rho_{SS} = \mathrm{id}_{\mathcal{E}(S)} : \mathcal{E}(S) \to \mathcal{E}(S)$ for all open $S \subseteq X$; and

(ii) $\eta_{SUV} \odot (\mathrm{id}_{\rho_{UV}} * \eta_{STU}) = \eta_{STV} \odot (\eta_{TUV} * \mathrm{id}_{\rho_{ST}}) :$
$\rho_{UV} \circ \rho_{TU} \circ \rho_{ST} \Longrightarrow \rho_{SV}$ for all open $V \subseteq U \subseteq T \subseteq S \subseteq X$.

Definition (Continued)

A prestack $\mathcal{E}$ on X is called a *stack* (or *stack in groupoids*, or 2-*sheaf*) on X if whenever $S \subseteq X$ is open and $\{T_i : i \in I\}$ is an open cover of S, then:

(iii) If $\alpha, \beta : A \to B$ are morphisms in $\mathcal{E}(S)$ and $\rho_{ST_i}(\alpha) = \rho_{ST_i}(\beta) : \rho_{ST_i}(A) \to \rho_{ST_i}(B)$ in $\mathcal{E}(T_i)$ for all $i \in I$, then $\alpha = \beta$.

(iv) If A, B are objects of $\mathcal{E}(S)$ and $\alpha_i : \rho_{ST_i}(A) \to \rho_{ST_i}(B)$ are morphisms in $\mathcal{E}(T_i)$ for all $i \in I$ with $\rho_{T_i(T_i \cap T_j)}(\alpha_i) = \rho_{T_j(T_i \cap T_j)}(\alpha_j)$ in $\mathcal{E}(T_i \cap T_j)$ for $i, j \in I$, there exists $\alpha : A \to B$ in $\mathcal{E}(S)$ (unique by (iii)) with $\rho_{ST_i}(\alpha) = \alpha_i$ for $i \in I$.

(v) If $A_i \in \mathcal{E}(T_i)$ for $i \in I$ and $\alpha_{ij} : \rho_{T_i(T_i \cap T_j)}(A_i) \to \rho_{T_j(T_i \cap T_j)}(A_j)$ are morphisms in $\mathcal{E}(T_i \cap T_j)$ for $i, j \in I$ with $\rho_{(T_j \cap T_k)(T_i \cap T_j \cap T_k)}(\alpha_{jk}) \circ \rho_{(T_i \cap T_j)(T_i \cap T_j \cap T_k)}(\alpha_{ij}) = \rho_{(T_i \cap T_k)(T_i \cap T_j \cap T_k)}(\alpha_{ik})$ for all $i, j, k \in I$, then there exist $A \in \mathcal{E}(S)$ and morphisms $\beta_i : A_i \to \rho_{ST_i}(A)$ for $i \in I$ such that $\rho_{T_i(T_i \cap T_j)}(\beta_i) = \rho_{T_j(T_i \cap T_j)}(\beta_j) \circ \alpha_{ij}$ for all $i, j \in I$.

In Theorem 6.6 we showed that coordinate changes of M-Kuranishi neighbourhoods have a sheaf property. There is an analogous stack property for coordinate changes of orbifold charts.

Theorem 7.11

Let $(V_i,\Gamma_i,\psi_i),(V_j,\Gamma_j,\psi_j)$ be orbifold charts on X. For each open $S\subseteq \mathrm{Im}\,\psi_i\cap\mathrm{Im}\,\psi_j$, write $\mathcal{C}oord\bigl((V_i,\Gamma_i,\psi_i),(V_j,\Gamma_j,\psi_j)\bigr)(S)$ for the groupoid of coordinate changes $(P_{ij},\pi_{ij},\phi_{ij}):(V_i,\Gamma_i,\psi_i)\rightarrow(V_j,\Gamma_j,\psi_j)$ over S, and for all open $T\subseteq S\subseteq \mathrm{Im}\,\psi_i\cap\mathrm{Im}\,\psi_j$ define

$$\rho_{ST}:\mathcal{C}oord\bigl((V_i,\Gamma_i,\psi_i),(V_j,\Gamma_j,\psi_j)\bigr)(S)\longrightarrow$$
$$\mathcal{C}oord\bigl((V_i,\Gamma_i,\psi_i),(V_j,\Gamma_j,\psi_j)\bigr)(T)\ \text{ by }\ \rho_{ST}=|_T,$$

and for all open $U\subseteq T\subseteq S\subseteq X$ define
$\eta_{STU}=\mathrm{id}_{\rho_{SU}}:\rho_{TU}\circ\rho_{ST}=\rho_{SU}\Rightarrow\rho_{SU}$.
Then $\mathcal{C}oord\bigl((V_i,\Gamma_i,\psi_i),(V_j,\Gamma_j,\psi_j)\bigr)$ is a stack on $\mathrm{Im}\,\psi_i\cap\mathrm{Im}\,\psi_j$.

The nontrivial part of this is a gluing result for principal Γ_j-bundles on a cover of V_i, with given isomorphisms on double overlaps and an associativity condition on triple overlaps.

7.4. The weak 2-category of orbifolds **Orb**

Definition 7.12

Let X be a Hausdorff, second countable topological space. An *orbifold structure* $\mathcal{O}$ *on* X *of dimension* $n\in\mathbb{N}$ is data $\mathcal{O}=\bigl(I,(V_i,\Gamma_i,\psi_i)_{i\in I},\ \Phi_{ij,\ i,j\in I},\ \lambda_{ijk,\ i,j,k\in I}\bigr)$, where:

(a) I is an indexing set.
(b) (V_i,Γ_i,ψ_i) is an orbifold chart on X for each $i\in I$, with $\dim V_i=n$. Write $S_i=\mathrm{Im}\,\psi_i$, $S_{ij}=\mathrm{Im}\,\psi_i\cap\mathrm{Im}\,\psi_j$, etc.
(c) $\Phi_{ij}=(P_{ij},\pi_{ij},\phi_{ij}):(V_i,\Gamma_i,\psi_i)|_{S_{ij}}\rightarrow(V_j,\Gamma_j,\psi_j)|_{S_{ij}}$ is a coordinate change for all $i,j\in I$.
(d) $\lambda_{ijk}:\Phi_{jk}\circ\Phi_{ij}|_{S_{ijk}}\Rightarrow\Phi_{ik}|_{S_{ijk}}$ is a 2-morphism for all $i,j,k\in I$.
(e) $\bigcup_{i\in I}\mathrm{Im}\,\psi_i=X$. (f) $\Phi_{ii}=\mathrm{id}_{(V_i,\Gamma_i,\psi_i)}$ for all $i\in I$.
(g) $\lambda_{iij}=\lambda_{ijj}=\mathrm{id}_{\Phi_{ij}}$ for all $i,j\in I$.
(h) $\lambda_{ikl}\odot(\mathrm{id}_{\Phi_{kl}}*\lambda_{ijk})|_{S_{ijkl}}=\lambda_{ijl}\odot(\lambda_{jkl}*\mathrm{id}_{\Phi_{ij}})|_{S_{ijkl}}:$ $\Phi_{kl}\circ\Phi_{jk}\circ\Phi_{ij}|_{S_{ijkl}}\Longrightarrow\Phi_{il}|_{S_{ijkl}}$ for all $i,j,k,l\in I$.

We call $\mathfrak{X}=(X,\mathcal{O})$ an *orbifold*, of *dimension* $\dim\mathfrak{X}=n$.

Recall that to define M-Kuranishi spaces in §6, which form a category, we specified data on S_i, S_{ij}, and imposed conditions on S_{ijk}. Here for orbifolds, which form a 2-category, we specify data on S_i, S_{ij}, S_{ijk}, and impose conditions on quadruple overlaps S_{ijkl}.

We call $\mathfrak{X}$ an *effective orbifold* if the orbifold charts (V_i, Γ_i, ψ_i) are effective, that is, if Γ_i acts (locally) effectively on V_i for all $i \in I$.

We can also define 1-morphisms and 2-morphisms of orbifolds. To do this, given a continuous map $f : X \to Y$ and orbifold charts (V_i, Γ_i, ψ_i), (W_j, Δ_j, χ_j) on X, Y, we have to define 1-morphisms $(P_{ij}, \pi_{ij}, f_{ij}) : (V_i, \Gamma_i, \psi_i) \to (W_j, \Delta_j, \chi_j)$ of orbifold charts over f, and 2-morphisms $\eta_{ij} : (P_{ij}, \pi_{ij}, f_{ij}) \Rightarrow (\tilde{P}_{ij}, \tilde{\pi}_{ij}, \tilde{f}_{ij})$, compositions $\circ, \odot, *$, and identities. These generalize Definitions 7.7 and 7.8 for $f = \mathrm{id}_X$, so we leave them as an exercise.

Definition 7.13

Let $\mathfrak{X} = (X, \mathcal{O})$ and $\mathfrak{Y} = (Y, \mathcal{P})$ be orbifolds, with $\mathcal{O} = \bigl(I, (V_i, \Gamma_i, \psi_i)_{i\in I},\ \Phi_{ii', \, i,i'\in I},\ \lambda_{ii'i'', \, i,i',i''\in I}\bigr)$ and $\mathcal{P} = \bigl(J, (W_j, \Delta_j, \chi_j)_{j\in J},\ \Psi_{jj', \, j,j'\in J},\ \mu_{jj'j'', \, j,j',j''\in J}\bigr)$. A *1-morphism* $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ is data $\mathfrak{f} = \bigl(f, \mathfrak{f}_{ij, \, i\in I, \, j\in J},\ F_{ii', \, i,i'\in I}^{j, \, j\in J},\ F_{i, \, i\in I}^{jj', \, j,j'\in J}\bigr)$, with:

(a) $f : X \to Y$ is a continuous map.

(b) $\mathfrak{f}_{ij} = (P_{ij}, \pi_{ij}, f_{ij}) : (V_i, \Gamma_i, \psi_i) \to (W_j, \Delta_j, \chi_j)$ is a 1-morphism of orbifold charts over f for all $i \in I$, $j \in J$.

(c) $F_{ii'}^j : \mathfrak{f}_{i'j} \circ \Phi_{ii'} \Rightarrow \mathfrak{f}_{ij}$ is a 2-morphism over f for $i, i' \in I$, $j \in J$.

(d) $F_i^{jj'} : \Psi_{jj'} \circ \mathfrak{f}_{ij} \Rightarrow \mathfrak{f}_{ij'}$ is a 2-morphism over f for $i \in I$, $j, j' \in J$.

(e) $F_{ii}^j = F_i^{jj} = \mathrm{id}_{\mathfrak{f}_{ij}}$.

(f) $F_{ii''}^j \odot (\mathrm{id}_{\mathfrak{f}_{i''j}} * \lambda_{ii'i''}) = F_{ii'}^j \odot (F_{i'i''}^j * \mathrm{id}_{\Phi_{ii'}}) : \mathfrak{f}_{i''j} \circ \Phi_{i'i''} \circ \Phi_{ii'} \Rightarrow \mathfrak{f}_{i''j}$.

(g) $F_i^{jj'} \odot (\mathrm{id}_{\Psi_{jj'}} * F_{ii'}^j) = F_{ii'}^{j'} \odot (F_{i'}^{jj'} * \mathrm{id}_{\Phi_{ii'}}) : \Psi_{jj'} \circ \mathfrak{f}_{i'j} \circ \Phi_{ii'} \Rightarrow \mathfrak{f}_{ij'}$.

(h) $F_i^{j'j''} \odot (\mathrm{id}_{\Psi_{j'j''}} * F_i^{jj'}) = F_i^{jj''} \odot (\mu_{jj'j''} * \mathrm{id}_{\mathfrak{f}_{ij}}) : \Psi_{j'j''} \circ \Psi_{jj'} \circ \mathfrak{f}_{ij} \Rightarrow \mathfrak{f}_{ij''}$.

Here (c)–(h) hold for all $i, j, \ldots$, restricted to appropriate domains.

Definition 7.14

Let $\mathfrak{f},\mathfrak{g}:\mathfrak{X}\to\mathfrak{Y}$ be 1-morphisms of orbifolds, with
$\mathfrak{f}=\bigl(f,\mathfrak{f}_{ij,\,i\in I,\,j\in J},\ F_{ii',\,i,i'\in I}^{j,\,j\in J},\ F_{i,\,i\in I}^{jj',\,j,j'\in J}\bigr)$,
$\mathfrak{g}=\bigl(g,\mathfrak{g}_{ij,\,i\in I,\,j\in J},\ G_{ii',\,i,i'\in I}^{j,\,j\in J},\ G_{i,\,i\in I}^{jj',\,j,j'\in J}\bigr)$. Suppose the continuous maps $f,g:X\to Y$ satisfy $f=g$. A *2-morphism* $\eta:\mathfrak{f}\Rightarrow\mathfrak{g}$ is data $\eta=(\eta_{ij,\,i\in I,\,j\in J})$, where $\eta_{ij}:\mathfrak{f}_{ij}\Rightarrow\mathfrak{g}_{ij}$ is a 2-morphism of orbifold charts over $f=g$, satisfying:

(a) $G_{ii'}^{j}\odot(\eta_{i'j}*\mathrm{id}_{\Phi_{ii'}})=\eta_{ij}\odot F_{ii'}^{j}:\mathfrak{f}_{i'j}\circ\Phi_{ii'}\Rightarrow\mathfrak{g}_{ij}$ for $i,i'\in I$, $j\in J$.

(b) $G_{i}^{jj'}\odot(\mathrm{id}_{\Psi_{jj'}}*\eta_{ij})=\eta_{ij'}\odot F_{i}^{jj'}:\Psi_{jj'}\circ\mathfrak{f}_{ij}\Rightarrow\mathfrak{g}_{ij'}$ for $i\in I$, $j,j'\in J$.

We can then define composition of 1- and 2-morphisms, identity 1- and 2-morphisms, and so on, making orbifolds into a weak 2-category **Orb**.

Composition of 1-morphisms $\mathfrak{g}\circ\mathfrak{f}$ is complicated: we have to use the analogue of the stack property Theorem 7.11 for 1-morphisms of orbifold charts to define $(\mathfrak{g}\circ\mathfrak{f})_{ik}$ in $\mathfrak{g}\circ\mathfrak{f}$. This only determines $(\mathfrak{g}\circ\mathfrak{f})_{ik}$ up to 2-isomorphism, so we have to make an arbitrary choice to define $\mathfrak{g}\circ\mathfrak{f}$. Because of this, we need not have $\mathfrak{h}\circ(\mathfrak{g}\circ\mathfrak{f})=(\mathfrak{h}\circ\mathfrak{g})\circ\mathfrak{f}$, instead we prove the existence of a natural 2-isomorphism $\alpha_{\mathfrak{h},\mathfrak{g},\mathfrak{f}}:\mathfrak{h}\circ(\mathfrak{g}\circ\mathfrak{f})\Rightarrow(\mathfrak{h}\circ\mathfrak{g})\circ\mathfrak{f}$. This is why **Orb** is a weak 2-category rather than a strict 2-category.

Note that the arguments used here are of two kinds. First in §7.1-§7.2 we use a lot of differential geometry to construct a 2-category of orbifold charts, and prove the stack property. But for the second part in §7.3, there is no differential geometry, we use 2-categories and stack theory to define the weak 2-category **Orb**.

To generalize to Kuranishi spaces, we first need to construct a 2-category of Kuranishi charts, and prove the stack property. The second part, construction of the weak 2-category **Kur** using 2-categories and stack theory, is standard, identical to **Orb**.

Kuranishi spaces

Lecture 8 of 14: Kuranishi spaces

Dominic Joyce, Oxford University
Summer 2015

These slides available at
`http://people.maths.ox.ac.uk/∼joyce/`

Plan of talk:

8. Kuranishi spaces

We now define a weak 2-category **Kur** of *Kuranishi spaces* $\mathbf{X}$, a kind of derived orbifold, following my arXiv:1409.6908, §4. One can also define derived orbifolds using C^∞-algebraic geometry by generalizing the definition of d-manifolds, replacing C^∞-schemes $\underline{X}$ by Deligne–Mumford C^∞-stacks $\mathcal{X}$. This yields a strict 2-category **dOrb** of *d-orbifolds*, as in my arXiv:1208.4948. There is an equivalence of weak 2-categories $\mathbf{Kur} \simeq \mathbf{dOrb}$, so Kuranishi spaces and d-orbifolds are interchangeable. Kuranishi spaces are simpler. The definition of Kuranishi spaces combines those of M-Kuranishi spaces in §6, and orbifolds in §7. We define 2-categories $\mathrm{Kur}_S(X)$ of 'Kuranishi neighbourhoods' on X supported on open $S \subseteq X$, with restriction functors $|_T : \mathrm{Kur}_S(X) \to \mathrm{Kur}_T(X)$ for open $T \subseteq S \subseteq X$, and show they satisfy the stack property. Then the same method as for orbifolds defines Kuranishi spaces as topological spaces with an atlas of Kuranishi neighbourhoods.

In fact 'Kuranishi spaces' (with a different, non-equivalent definition, which we will call 'FOOO Kuranishi spaces') have been used for many years in the work of Fukaya et al. in symplectic geometry (Fukaya and Ono 1999, Fukaya–Oh–Ohta–Ono 2009), as the geometric structure on moduli spaces of J-holomorphic curves. There are problems with their theory (e.g. there is no notion of morphism of FOOO Kuranishi space), and I claim my definition is the 'correct' definition of Kuranishi space, which should replace the FOOO definition. Any FOOO Kuranishi space $\mathbf{X}$ can be made into a Kuranishi space $\mathbf{X}'$ in my sense, uniquely up to equivalence in **Kur**, so this replacement can be done fairly painlessly.

8.1. Kuranishi neighbourhoods and coordinate changes

Definition 8.1

Let X be a topological space. A *Kuranishi neighbourhood* on X is a quintuple (V, E, Γ, s, ψ) such that:

(a) V is a smooth manifold.

(b) $\pi : E \to V$ is a vector bundle over V, the *obstruction bundle*.

(c) Γ is a finite group with compatible smooth actions on V and E preserving the vector bundle structure.

(d) $s : V \to E$ is a Γ-equivariant smooth section of E, the *Kuranishi section*.

(e) $\psi : s^{-1}(0)/\Gamma \to X$ is a homeomorphism with an open $\mathrm{Im}\,\psi \subseteq X$. We write $\bar\psi$ for the composition $s^{-1}(0) \to s^{-1}(0)/\Gamma \xrightarrow{\psi} X$.

If $S \subseteq X$ is open, we call (V, E, Γ, s, ψ) a *Kuranishi neighbourhood over* S if $S \subseteq \mathrm{Im}\,\psi \subseteq X$.

This is the same as Fukaya-Oh-Ohta-Ono Kuranishi neighbourhoods.

Definition 8.2

Let X be a topological space, $(V_i, E_i, \Gamma_i, s_i, \psi_i)$, $(V_j, E_j, \Gamma_j, s_j, \psi_j)$ be Kuranishi neighbourhoods on X, and $S \subseteq \mathrm{Im}\,\psi_i \cap \mathrm{Im}\,\psi_j \subseteq X$ be open. A 1-*morphism* $\Phi_{ij} : (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (V_j, E_j, \Gamma_j, s_j, \psi_j)$ *over* S is a quadruple $\Phi_{ij} = (P_{ij}, \pi_{ij}, \phi_{ij}, \hat\phi_{ij})$ satisfying:

(a) P_{ij} is a manifold with a smooth action of $\Gamma_i \times \Gamma_j$, with the Γ_j-action free.

(b) $\pi_{ij} : P_{ij} \to V_i$ is Γ_i-equivariant, Γ_j-invariant, and étale. The image $V_{ij} := \pi_{ij}(P_{ij})$ is a Γ_i-invariant open neighbourhood of $\bar\psi_i^{-1}(S)$ in V_i, and $\pi_{ij} : P_{ij} \to V_{ij}$ is a principal Γ_j-bundle.

(c) $\phi_{ij} : P_{ij} \to V_j$ is a Γ_i-invariant and Γ_j-equivariant smooth map.

(d) $\hat\phi_{ij} : \pi_{ij}^*(E_i) \to \phi_{ij}^*(E_j)$ is a $\Gamma_i \times \Gamma_j$-equivariant morphism of vector bundles on P_{ij}, using the given Γ_i-action and the trivial Γ_j-action on E_i, and vice versa for E_j.

(e) $\hat\phi_{ij}(\pi_{ij}^*(s_i)) = \phi_{ij}^*(s_j) + O(\pi_{ij}^*(s_i)^2)$.

(f) $\bar\psi_i \circ \pi_{ij} = \bar\psi_j \circ \phi_{ij}$ on $\pi_{ij}^{-1}(s_i^{-1}(0)) \subseteq P_{ij}$.

Here $[V_{ij}/\Gamma_i]\subseteq[V_i/\Gamma_i]$ is an open sub-orbifold, and $\phi_{ij}=(P_{ij},\pi_{ij},\phi_{ij}):[V_{ij}/\Gamma_i]\to[V_j/\Gamma_j]$ is a Hilsum–Skandalis morphism of orbifolds, as in §7.1. We can interpret E_i,E_j as *orbifold vector bundles* over $[V_i/\Gamma_i],[V_j/\Gamma_j]$ with sections s_i,s_j, and $\hat\phi_{ij}$ as a morphism $E_i\to\phi_{ij}^*(E_j)$ of vector bundles on $[V_{ij}/\Gamma_i]$ with $\hat\phi_{ij}(s_i)=\phi_{ij}^*(s_j)+O(s_i^2)$.

Thus, $(P_{ij},\pi_{ij},\phi_{ij},\hat\phi_{ij})$ is the orbifold analogue of the morphisms $[V_{ij},\phi_{ij},\hat\phi_{ij}]:(V_i,E_i,s_i,\psi_i)\to(V_j,E_j,s_j,\psi_j)$ of M-Kuranishi neighbourhoods in §6.1, with $(P_{ij},\pi_{ij},\phi_{ij})$ in place of ϕ_{ij}.

For M-Kuranishi spaces, we took equivalence classes $[V_{ij},\phi_{ij},\hat\phi_{ij}]$ of triples $(V_{ij},\phi_{ij},\hat\phi_{ij})$. Here we do not take equivalence classes for 1-morphisms, but we will for 2-morphisms.

Definition 8.3

Let $\Phi_{ij},\Phi'_{ij}:(V_i,E_i,\Gamma_i,s_i,\psi_i)\to(V_j,E_j,\Gamma_j,s_j,\psi_j)$ be 1-morphisms of Kuranishi neighbourhoods over $S\subseteq\operatorname{Im}\psi_i\cap\operatorname{Im}\psi_j\subseteq X$, where $\Phi_{ij}=(P_{ij},\pi_{ij},\phi_{ij},\hat\phi_{ij})$ and $\Phi'_{ij}=(P'_{ij},\pi'_{ij},\phi'_{ij},\hat\phi'_{ij})$.

Consider triples $(\dot P_{ij},\lambda_{ij},\hat\lambda_{ij})$ satisfying:

(a) $\dot P_{ij}$ is a $\Gamma_i\times\Gamma_j$-invariant open neighbhd of $\pi_{ij}^{-1}(\bar\psi_i^{-1}(S))$ in P_{ij}.

(b) $\lambda_{ij}:\dot P_{ij}\to P'_{ij}$ is a $\Gamma_i\times\Gamma_j$-equivariant smooth map with $\pi'_{ij}\circ\lambda_{ij}=\pi_{ij}|_{\dot P_{ij}}$. This implies that λ_{ij} is a diffeomorphism with a $\Gamma_i\times\Gamma_j$-invariant open set $\lambda_{ij}(\dot P_{ij})$ in P'_{ij}.

(c) $\hat\lambda_{ij}:\pi_{ij}^*(E_i)|_{\dot P_{ij}}\to\phi_{ij}^*(TV_j)|_{\dot P_{ij}}$ is a Γ_i- and Γ_j-invariant smooth morphism of vector bundles on $\dot P_{ij}$, satisfying

$$\begin{aligned}&\phi'_{ij}\circ\lambda_{ij}=\phi_{ij}|_{\dot P_{ij}}+\hat\lambda_{ij}\cdot\pi_{ij}^*(s_i)+O\bigl(\pi_{ij}^*(s_i)^2\bigr)\quad\text{and}\\&\lambda_{ij}^*(\hat\phi'_{ij})=\hat\phi_{ij}|_{\dot P_{ij}}+\hat\lambda_{ij}\cdot\phi_{ij}^*(\mathrm{d}s_j)+O\bigl(\pi_{ij}^*(s_i)\bigr)\quad\text{on }\dot P_{ij}.\end{aligned}\tag{8.1}$$

Definition (Continued)

Define an equivalence relation $\approx$ (or $\approx_S$) on such triples by $(\dot P_{ij},\lambda_{ij},\hat\lambda_{ij})\approx(\dot P'_{ij},\lambda'_{ij},\hat\lambda'_{ij})$ if there exists an open neighbourhood $\ddot P_{ij}$ of $\pi_{ij}^{-1}(\bar\psi_i^{-1}(S))$ in $\dot P_{ij}\cap\dot P'_{ij}$ with

$$\lambda_{ij}|_{\ddot P_{ij}}=\lambda'_{ij}|_{\ddot P_{ij}} \quad\text{and}\quad \hat\lambda_{ij}|_{\ddot P_{ij}}=\hat\lambda'_{ij}|_{\ddot P_{ij}}+O\bigl(\pi_{ij}^*(s_i)\bigr) \quad\text{on } \ddot P_{ij}. \quad (8.2)$$

Write $[\dot P_{ij},\lambda_{ij},\hat\lambda_{ij}]$ for the $\approx$-equivalence class of $(\dot P_{ij},\lambda_{ij},\hat\lambda_{ij})$. We say that $[\dot P_{ij},\lambda_{ij},\hat\lambda_{ij}]:\Phi_{ij}\Rightarrow\Phi'_{ij}$ is a *2-morphism of 1-morphisms of Kuranishi neighbourhoods on X over S*, or just a *2-morphism over S*. We often write $\Lambda_{ij}=[\dot P_{ij},\lambda_{ij},\hat\lambda_{ij}]$.

Here (8.1) is the orbifold version of standard model 2-morphisms of d-manifolds in §5.3, and (8.2) the orbifold version of when two standard model 2-morphisms are equal from Theorem 5.3(b).

A 2-morphism $[\dot P_{ij},\lambda_{ij},\hat\lambda_{ij}]:\Phi_{ij}\Rightarrow\Phi'_{ij}$ really consists of three pieces of data:

(i) An open neighbourhood $[\dot V_{ij}/\Gamma_i]$ of $\psi_i^{-1}(S)$ in $[V_{ij}/\Gamma_i]\cap[V'_{ij}/\Gamma_i]\subseteq[V_i/\Gamma_i]$, where $\dot P_{ij}=\pi_{ij}^{-1}(\dot V_{ij})$.

(ii) A 2-morphism of orbifolds $\lambda_{ij}:(P_{ij},\phi_{ij},\phi_{ij})|_{[\dot V_{ij}/\Gamma_i]}\Rightarrow(P'_{ij},\phi'_{ij},\phi'_{ij})|_{[\dot V_{ij}/\Gamma_i]}$, in the sense of §7.

(iii) A 'standard model' 2-morphism of derived manifolds $\hat\lambda_{ij}$, lifted to derived orbifolds.

There is little interaction between (ii) and (iii); the 'orbifold' and 'derived manifold' generalizations of manifolds are more-or-less independent.

We can define composition of 1- and 2-morphisms of Kuranishi neighbourhoods, and identity 1- and 2-morphisms, by combining the orbifold story in §7 with the derived manifold story in §5–§6. In this way we obtain a strict 2-category $\mathrm{Kur}_S(X)$ of Kuranishi neighbourhoods over $S \subseteq X$.

If $T \subseteq S \subseteq X$ are open there is a restriction 2-functor $|_T : \mathrm{Kur}_S(X) \to \mathrm{Kur}_T(X)$. On objects $(V_i, E_i, \Gamma_i, s_i, \psi_i)$ and 1-morphisms $(P_{ij}, \pi_{ij}, \phi_{ij}, \hat{\phi}_{ij})$, this just acts as the identity. But for 2-morphisms $[\dot{P}_{ij}, \lambda_{ij}, \hat{\lambda}_{ij}]$ in $\mathrm{Kur}_S(X)$, the equivalence relation $\approx_S$ on triples $(\dot{P}_{ij}, \lambda_{ij}, \hat{\lambda}_{ij})$ depends on S, as (8.2) must hold in a neighbourhood of $\pi_{ij}^{-1}(\bar{\psi}_i^{-1}(S))$. So $|_T$ maps the $\approx_S$-equivalence class $[\dot{P}_{ij}, \lambda_{ij}, \hat{\lambda}_{ij}]_S$ to the $\approx_T$-equivalence class $[\dot{P}_{ij}, \lambda_{ij}, \hat{\lambda}_{ij}]_T$.

Definition 8.4

A 1-morphism $\Phi_{ij} : (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (V_j, E_j, \Gamma_j, s_j, \psi_j)$ of Kuranishi neighbourhoods over $S \subseteq X$ is a *coordinate change* over S if it is an equivalence in the 2-category $\mathrm{Kur}_S(X)$.

Theorems 5.7 and 6.5 gave criteria for a standard model 1-morphism to be an equivalence, and a morphism of M-Kuranishi neighbourhoods to be a coordinate change. Here is the orbifold analogue:

Theorem 8.5

Let $\Phi_{ij} = (P_{ij}, \pi_{ij}, \phi_{ij}, \hat{\phi}_{ij}) : (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (V_j, E_j, \Gamma_j, s_j, \psi_j)$ be a 1-morphism of Kuranishi neighbourhoods over $S \subseteq X$. Let $p \in \pi_{ij}^{-1}(\bar{\psi}_i^{-1}(S)) \subseteq P_{ij}$, and set $v_i = \pi_{ij}(p) \in V_i$ and $v_j = \phi_{ij}(p) \in V_j$. Consider the complex of real vector spaces:

$$0 \to T_{v_i} V_i \xrightarrow{\mathrm{d}s_i|_{v_i} \oplus (\mathrm{d}\phi_{ij}|_p \circ \mathrm{d}\pi_{ij}|_p^{-1})} E_i|_{v_i} \oplus T_{v_j} V_j \xrightarrow{-\hat{\phi}_{ij}|_p \oplus \mathrm{d}s_j|_{v_j}} E_j|_{v_j} \to 0. \quad (8.3)$$

Also consider the morphism of finite groups

$$\rho_p : \{(\gamma_i, \gamma_j) \in \Gamma_i \times \Gamma_j : (\gamma_i, \gamma_j) \cdot p = p\} \longrightarrow \{\gamma_j \in \Gamma_j : \gamma_j \cdot v_j = v_j\},$$
$$\rho_p : (\gamma_i, \gamma_j) \longmapsto \gamma_j. \quad (8.4)$$

Then Φ_{ij} is a coordinate change over S iff (8.3) is exact and (8.4) is an isomorphism for all $p \in \pi_{ij}^{-1}(\bar{\psi}_i^{-1}(S))$.

Example 8.6

In Fukaya–Oh–Ohta–Ono Kuranishi spaces, a 'coordinate change' $(V_{ij},\rho_{ij},\varphi_{ij},\hat\varphi_{ij}):(V_i,E_i,\Gamma_i,s_i,\psi_i)\rightarrow(V_j,E_j,\Gamma_j,s_j,\psi_j)$ consists of a Γ_i-invariant open $V_{ij}\subseteq V_i$, a group morphism $\rho_{ij}:\Gamma_i\rightarrow\Gamma_j$, a ρ_{ij}-equivariant embedding of submanifolds $\varphi_{ij}:V_{ij}\hookrightarrow V_j$, and a ρ_{ij}-equivariant embedding of vector bundles $\hat\varphi_{ij}:E_i\vert_{V_{ij}}\hookrightarrow\varphi_{ij}^*(E_j)$ with $\hat\varphi_{ij}\circ s_i=\varphi_{ij}^*(s_j)$, such that the induced morphism $(\mathrm{d}s_i)_*:\varphi_{ij}^*(TV_j)/TV_{ij}\rightarrow\varphi_{ij}^*(E_j)/E_i$ is an isomorphism near $s_i^{-1}(0)$, and ρ restricts to an isomorphism $\mathrm{Stab}_{\Gamma_i}(v)\rightarrow\mathrm{Stab}_{\Gamma_j}(\varphi_{ij}(v))$ for all $v\in\bar\psi_i^{-1}(S)$.
By Theorem 8.5 we can show that this induces a coordinate change $(P_{ij},\pi_{ij},\phi_{ij},\hat\phi_{ij})$ in our sense, with $P_{ij}=V_i\times\Gamma_j$ the trivial principal Γ_j-bundle over V_i. But FOOO coordinate changes are very special examples of ours; they only exist if $\dim V_i\leqslant\dim V_j$. Our coordinate changes are more flexible, and are invertible up to 2-isomorphisms.

As for Theorems 6.6 and 7.11 we have:

Theorem 8.7

Let $(V_i,E_i,\Gamma_i,s_i,\psi_i),(V_j,E_j,\Gamma_j,s_j,\psi_j)$ be Kuranishi neighbourhoods on X. For each open $S\subseteq\mathrm{Im}\,\psi_i\cap\mathrm{Im}\,\psi_j$, write $\mathcal{C}oord\bigl((V_i,E_i,\Gamma_i,s_i,\psi_i),(V_j,E_j,\Gamma_j,s_j,\psi_j)\bigr)(S)$ for the groupoid of coordinate changes $(P_{ij},\pi_{ij},\phi_{ij},\hat\phi_{ij}):(V_i,E_i,\Gamma_i,s_i,\psi_i)\rightarrow(V_j,E_j,\Gamma_j,s_j,\psi_j)$ over S, and for all open $T\subseteq S\subseteq\mathrm{Im}\,\psi_i\cap\mathrm{Im}\,\psi_j$ define

$$\rho_{ST}:\mathcal{C}oord\bigl((V_i,E_i,\Gamma_i,s_i,\psi_i),(V_j,E_j,\Gamma_j,s_j,\psi_j)\bigr)(S)\longrightarrow \mathcal{C}oord\bigl((V_i,E_i,\Gamma_i,s_i,\psi_i),(V_j,E_j,\Gamma_j,s_j,\psi_j)\bigr)(T)\ \text{by}\ \rho_{ST}=\vert_T,$$

and for all open $U\subseteq T\subseteq S\subseteq X$ define $\eta_{STU}=\mathrm{id}_{\rho_{SU}}:\rho_{TU}\circ\rho_{ST}=\rho_{SU}\Rightarrow\rho_{SU}$.
Then $\mathcal{C}oord\bigl((V_i,E_i,\Gamma_i,s_i,\psi_i),(V_j,E_j,\Gamma_j,s_j,\psi_j)\bigr)$ is a **stack** *on the topological space $\mathrm{Im}\,\psi_i\cap\mathrm{Im}\,\psi_j$.*

The important, nontrivial part is a gluing result for coordinate changes on an open cover, with given 2-isomorphisms on double overlaps and an associativity condition on triple overlaps.

8.2. The 2-category of Kuranishi spaces **Kur**

Definition 8.8

Let X be a Hausdorff, second countable topological space. A *Kuranishi structure* $\mathcal{K}$ on X of *virtual dimension* $n \in \mathbb{N}$ is data $\mathcal{K} = \bigl(I, (V_i, E_i, \Gamma_i, s_i, \psi_i)_{i\in I}, \Phi_{ij,\ i,j\in I}, \Lambda_{ijk,\ i,j,k\in I}\bigr)$, where:

(a) I is an indexing set.

(b) $(V_i, E_i, \Gamma_i, s_i, \psi_i)$ is a Kuranishi neighbourhood on X for $i \in I$, with $\dim V_i - \operatorname{rank} E_i = n$. Write $S_{ij} = \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$, etc.

(c) $\Phi_{ij} = (P_{ij}, \pi_{ij}, \phi_{ij}, \hat\phi_{ij}) : (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (V_j, E_j, \Gamma_j, s_j, \psi_j)$ is a coordinate change over S_{ij} for $i, j \in I$.

(d) $\Lambda_{ijk} : \Phi_{jk} \circ \Phi_{ij} \Rightarrow \Phi_{ik}$ is a 2-morphism over S_{ijk} for $i, j, k \in I$.

(e) $\bigcup_{i\in I} \operatorname{Im}\psi_i = X$. (f) $\Phi_{ii} = \mathrm{id}_{(V_i, E_i, \Gamma_i, s_i, \psi_i)}$ for $i \in I$.

(g) $\Lambda_{iij} = \Lambda_{ijj} = \mathrm{id}_{\Phi_{ij}}$ for $i, j \in I$.

(h) $\Lambda_{ikl} \odot (\mathrm{id}_{\Phi_{kl}} * \Lambda_{ijk})|_{S_{ijkl}} = \Lambda_{ijl} \odot (\Lambda_{jkl} * \mathrm{id}_{\Phi_{ij}})|_{S_{ijkl}} :$
$\Phi_{kl} \circ \Phi_{jk} \circ \Phi_{ij}|_{S_{ijkl}} \Longrightarrow \Phi_{il}|_{S_{ijkl}}$ for $i, j, k, l \in I$.

We call $\mathbf{X} = (X, \mathcal{K})$ a *Kuranishi space*, with $\operatorname{vdim}\mathbf{X} = n$.

Definition 8.8 is a direct analogue of orbifolds in Definition 7.12, replacing orbifold charts by Kuranishi neighbourhoods. It covers X by an 'atlas of charts' $(V_i, E_i, \Gamma_i, s_i, \psi_i)$ over $\operatorname{Im}\psi_i \subseteq X$, with coordinate changes Φ_{ij} on double overlaps $\operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$, and 2-isomorphisms $\Lambda_{ijk} : \Phi_{jk} \circ \Phi_{ij}|_{S_{ijk}} \Rightarrow \Phi_{ik}|_{S_{ijk}}$ on triple overlaps $\operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j \cap \operatorname{Im}\psi_k$, with associativity $\Lambda_{ikl} \odot (\mathrm{id}_{\Phi_{kl}} * \Lambda_{ijk})|_{S_{ijkl}} = \Lambda_{ijl} \odot (\Lambda_{jkl} * \mathrm{id}_{\Phi_{ij}})|_{S_{ijkl}}$ on quadruple overlaps $\operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j \cap \operatorname{Im}\psi_k \cap \operatorname{Im}\psi_l$.

Once you have grasped the idea that Kuranishi neighbourhoods over $S \subseteq X$ form a 2-category, with restriction $|_T$ to open subsets $T \subseteq S \subseteq X$, Definition 8.8, although complicated, is obvious, and necessary: it is the only sensible way to make a global space by gluing local charts in the world of 2-categories.

We can also define 1-morphisms and 2-morphisms of Kuranishi spaces. To do this, given a continuous map $f : X \to Y$ and Kuranishi neighbourhoods $(V_i, E_i, \Gamma_i, s_i, \psi_i)$, $(W_j, F_j, \Delta_j, t_j, \chi_j)$ on X, Y, and open $S \subseteq \operatorname{Im} \psi_i \cap f^{-1}(\operatorname{Im} \chi_j)$, we first have to define 1-morphisms $\mathbf{f}_{ij} = (P_{ij}, \pi_{ij}, f_{ij}, \hat{f}_{ij}) : (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (W_j, F_j, \Delta_j, t_j, \chi_j)$ of Kuranishi neighbourhoods over S and f, and 2-morphisms $\Lambda_{ij} : (P_{ij}, \pi_{ij}, f_{ij}, \hat{f}_{ij}) \Rightarrow (P'_{ij}, \pi'_{ij}, f'_{ij}, \hat{f}'_{ij})$, compositions $\circ, \odot, *$, and identities. These generalize Definitions 8.2 and 8.3 for $f = \mathrm{id}_X$, so we leave them as an exercise.

Definition 8.9

Let $\mathbf{X} = (X, \mathcal{K})$ and $\mathbf{Y} = (Y, \mathcal{L})$ be Kuranishi spaces, with $\mathcal{K} = \bigl(I, (V_i, E_i, \Gamma_i, s_i, \psi_i)_{i\in I}, \Phi_{ii',\, i,i'\in I}, \Lambda_{ii'i'',\, i,i',i''\in I}\bigr)$ and $\mathcal{L} = \bigl(J, (W_j, F_j, \Delta_j, t_j, \chi_j)_{j\in J}, \Psi_{jj',\, j,j'\in J}, \mathrm{M}_{jj'j'',\, j,j',j''\in J}\bigr)$. A *1-morphism* $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ is $\mathbf{f} = \bigl(f, \mathbf{f}_{ij,\, i\in I,\, j\in J}, F^{j,\, j\in J}_{ii',\, i,i'\in I}, F^{jj',\, j,j'\in J}_{i,\, i\in I}\bigr)$, with: (a) $f : X \to Y$ is a continuous map.

(b) $\mathbf{f}_{ij} = (P_{ij}, \pi_{ij}, f_{ij}, \hat{f}_{ij}) : (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (W_j, F_j, \Delta_j, t_j, \chi_j)$ is a 1-morphism of Kuranishi neighbourhoods over $S = \operatorname{Im} \psi_i \cap f^{-1}(\operatorname{Im} \chi_j)$ and f for $i \in I$, $j \in J$.

(c) $F^j_{ii'} : \mathbf{f}_{i'j} \circ \Phi_{ii'} \Rightarrow \mathbf{f}_{ij}$ is a 2-morphism over f for $i, i' \in I$, $j \in J$.

(d) $F^{jj'}_i : \Psi_{jj'} \circ \mathbf{f}_{ij} \Rightarrow \mathbf{f}_{ij'}$ is a 2-morphism over f for $i \in I$, $j, j' \in J$.

(e) $F^j_{ii} = F^{jj}_i = \mathrm{id}_{\mathbf{f}_{ij}}$.

(f) $F^j_{ii''} \odot (\mathrm{id}_{\mathbf{f}_{i''j}} * \Lambda_{ii'i''}) = F^j_{ii'} \odot (F^j_{i'i''} * \mathrm{id}_{\Phi_{ii'}}) : \mathbf{f}_{i''j} \circ \Phi_{i'i''} \circ \Phi_{ii'} \Rightarrow \mathbf{f}_{i''j}$.

(g) $F^{jj'}_i \odot (\mathrm{id}_{\Psi_{jj'}} * F^j_{ii'}) = F^{j'}_{ii'} \odot (F^{jj'}_{i'} * \mathrm{id}_{\Phi_{ii'}}) : \Psi_{jj'} \circ \mathbf{f}_{i'j} \circ \Phi_{ii'} \Rightarrow \mathbf{f}_{ij'}$.

(h) $F^{j'j''}_i \odot (\mathrm{id}_{\Psi_{j'j''}} * F^{jj'}_i) = F^{jj''}_i \odot (\mathrm{M}_{jj'j''} * \mathrm{id}_{\mathbf{f}_{ij}}) : \Psi_{j'j''} \circ \Psi_{jj'} \circ \mathbf{f}_{ij} \Rightarrow \mathbf{f}_{ij''}$.

Here (c)–(h) hold for all $i,j,\ldots$, restricted to appropriate domains.

Definition 8.10

Let $\mathbf{f},\mathbf{g}:\mathbf{X}\to\mathbf{Y}$ be 1-morphisms of Kuranishi spaces, with
$\mathbf{f}=\bigl(f,\mathbf{f}_{ij,\; i\in I,\; j\in J},\; F_{ii',\; i,i'\in I}^{j,\; j\in J},\; F_{i,\; i\in I}^{jj',\; j,j'\in J}\bigr)$,
$\mathbf{g}=\bigl(g,\mathbf{g}_{ij,\; i\in I,\; j\in J},\; G_{ii',\; i,i'\in I}^{j,\; j\in J},\; G_{i,\; i\in I}^{jj',\; j,j'\in J}\bigr)$. Suppose the continuous maps $f,g:X\to Y$ satisfy $f=g$. A *2-morphism* $\Lambda:\mathbf{f}\Rightarrow\mathbf{g}$ is data $\Lambda=(\Lambda_{ij,\; i\in I,\; j\in J})$, where $\Lambda_{ij}:\mathbf{f}_{ij}\Rightarrow\mathbf{g}_{ij}$ is a 2-morphism of orbifold charts over $f=g$, satisfying:

(a) $G_{ii'}^{j}\odot(\Lambda_{i'j}*\mathrm{id}_{\Phi_{ii'}})=\Lambda_{ij}\odot F_{ii'}^{j}:\mathbf{f}_{i'j}\circ\Phi_{ii'}\Rightarrow\mathbf{g}_{ij}$ for $i,i'\in I$, $j\in J$.

(b) $G_{i}^{jj'}\odot(\mathrm{id}_{\Psi_{jj'}}*\Lambda_{ij})=\Lambda_{ij'}\odot F_{i}^{jj'}:\Psi_{jj'}\circ\mathbf{f}_{ij}\Rightarrow\mathbf{g}_{ij'}$ for $i\in I$, $j,j'\in J$.

Definitions 8.9, 8.10 are direct analogues of the orbifold versions. We can then define composition of 1- and 2-morphisms, identity 1- and 2-morphisms, and so on, making Kuranishi spaces into a weak 2-category **Kur**. Composition of 1-morphisms needs the stack property of 1-morphisms of Kuranishi neighbhds, as in Theorem 8.7.

Write $\mathbf{Kur_{tr\Gamma}}$ for the full 2-subcategory of **Kur** with objects $\mathbf{X}=(X,\mathcal{K})$ in which the Kuranishi neighbourhoods $(V_i,E_i,\Gamma_i,s_i,\psi_i)$ in $\mathcal{K}$ have $\Gamma_i=\{1\}$ for all $i\in I$. Write $\mathbf{Kur_{trG}}$ for the full 2-subcategory of $\mathbf{X}$ in which Γ_i acts freely on $s_i^{-1}(0)\subseteq V_i$ for all $i\in I$. Then $\mathbf{Kur_{tr\Gamma}}\subset\mathbf{Kur_{trG}}\subset\mathbf{Kur}$. Both $\mathbf{Kur_{tr\Gamma}},\mathbf{Kur_{trG}}$ are 2-categories of derived manifolds. In the notation of next time, $\mathbf{Kur_{trG}}$ is the full 2-subcategory of Kuranishi spaces $\mathbf{X}$ with trivial orbifold groups $\mathrm{Iso}_{\mathbf{X}}(x)=\{1\}$ for all $x\in\mathbf{X}$.

Theorem 8.11

There are equivalences of weak 2-categories

$$\mathbf{Kur_{tr\Gamma}}\simeq\mathbf{Kur_{trG}}\simeq\mathbf{dMan},\qquad \mathbf{Kur}\simeq\mathbf{dOrb},$$

and equivalences of (homotopy) categories

$$\mathbf{MKur}\simeq\mathrm{Ho}(\mathbf{Kur_{tr\Gamma}})\simeq\mathrm{Ho}(\mathbf{Kur_{trG}})\simeq\mathrm{Ho}(\mathbf{dMan}).$$

So for most purposes Kuranishi spaces (with trivial orbifold groups) and d-orbifolds (or d-manifolds) are interchangeable.

Derived Differential Geometry

Lecture 9 of 14: Differential Geometry of derived manifolds and orbifolds

Dominic Joyce, Oxford University
Summer 2015

These slides, and references, etc., available at
http://people.maths.ox.ac.uk/∼joyce/DDG2015

Plan of talk:

9. Differential Geometry of derived manifolds and orbifolds

Here are some important topics in ordinary differential geometry:

- Immersions, embeddings, and submanifolds.
  The Whitney Embedding Theorem.
- Submersions.
- Orientations.
- Transverse fibre products.
- Manifolds with boundary and corners.
- (Oriented) bordism groups.
- Fundamental classes of compact oriented manifolds in homology.

The next four lectures will explain how all these extend to derived manifolds and orbifolds.

9.1. Orbifold groups, tangent and obstruction spaces for d-orbifolds and Kuranishi spaces

In §5.4 and §6.3 we explained that a d-manifold / M-Kuranishi space $\mathbf{X}$ has functorial tangent spaces $T_x\mathbf{X}$ and obstruction spaces $O_x\mathbf{X}$ for $x \in \mathbf{X}$. We now discuss the orbifold versions of these, which are not quite functorial. Let $\mathbf{X}$ be a d-orbifold or Kuranishi space, and $x \in \mathbf{X}$. Then we can define the *orbifold group* $G_x\mathbf{X}$, a finite group, and the *tangent space* $T_x\mathbf{X}$ and *obstruction space* $O_x\mathbf{X}$, both finite-dimensional real representations of $G_x\mathbf{X}$. If (V, E, Γ, s, ψ) is a Kuranishi neighbourhood on $\mathbf{X}$ with $x \in \mathrm{Im}\,\psi$, and $v \in s^{-1}(0) \subseteq V$ with $\psi(v\Gamma) = x$ then we may write

$$\begin{aligned} G_x\mathbf{X} &= \mathrm{Stab}_\Gamma(v) = \{\gamma \in \Gamma : \gamma \cdot v = v\}, \\ T_x\mathbf{X} &= \mathrm{Ker}\big(\mathrm{d}s|_v : T_vV \to E|_v\big), \\ O_x\mathbf{X} &= \mathrm{Coker}\big(\mathrm{d}s|_v : T_vV \to E|_v\big). \end{aligned} \tag{9.1}$$

For d-manifolds, $T_x\mathbf{X}, O_x\mathbf{X}$ are unique up to canonical isomorphism, so we can treat them as unique. For d-orbifolds (and classical orbifolds), things are more subtle: $G_x\mathbf{X}, T_x\mathbf{X}, O_x\mathbf{X}$ are unique *up to isomorphism*, but *not up to canonical isomorphism*. That is, to define $G_x\mathbf{X}, T_x\mathbf{X}, O_x\mathbf{X}$ in (9.1) we had to *choose* $v \in s^{-1}(0)$ with $\psi(v) = x$.
If v' is an alternative choice yielding $G_x\mathbf{X}', T_x\mathbf{X}', O_x\mathbf{X}'$, then $v' = \delta \cdot v$ for some $\delta \in \Gamma$, and we have isomorphisms

$$\begin{aligned} G_x\mathbf{X} &\longrightarrow G_x\mathbf{X}', & \gamma &\longmapsto \delta\gamma\delta^{-1}, \\ T_x\mathbf{X} &\longrightarrow T_x\mathbf{X}', & t &\longmapsto T_v\delta(t), \\ O_x\mathbf{X} &\longrightarrow O_x\mathbf{X}', & o &\longmapsto \hat{T}_v\delta(o), \end{aligned}$$

where $T_v\delta : T_vV \to T_{\delta\cdot v}V$, $\hat{T}_v\delta : E|_v \to E|_{\delta\cdot v}$ are induced by the Γ-actions on V, E. But these isomorphisms are not unique, as we could replace δ by $\delta\epsilon$ for any $\epsilon \in G_x\mathbf{X}$.

Differential Geometry of derived manifolds and orbifolds
Fibre products of derived manifolds and orbifolds
Orbifold groups, tangent and obstruction spaces
Immersions, embeddings and d-submanifolds
Embedding derived manifolds into manifolds
Submersions
Orientations

Because of this, $G_x\mathbf{X}$ is canonical up to conjugation, i.e. up to automorphisms $G_x\mathbf{X} \to G_x\mathbf{X}$ of the form $\gamma \mapsto \epsilon\gamma\epsilon^{-1}$ for $\epsilon \in G_x\mathbf{X}$, and similarly $T_x\mathbf{X}, O_x\mathbf{X}$ are canonical up to the action of elements of $G_x\mathbf{X}$, i.e. up to automorphisms $T_x\mathbf{X} \to T_x\mathbf{X}$ mapping $t \mapsto \epsilon \cdot t$.
Our solution is to use the Axiom of Choice to *choose* an allowed triple $(G_x\mathbf{X}, T_x\mathbf{X}, O_x\mathbf{X})$ for all derived orbifolds $\mathbf{X}$ and $x \in \mathbf{X}$.
Similarly, for any 1-morphism $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ of derived orbifolds and $x \in \mathbf{X}$ with $\mathbf{f}(x) = y \in \mathbf{Y}$, we can define a group morphism $G_x\mathbf{f} : G_x\mathbf{X} \to G_y\mathbf{Y}$, and $G_x\mathbf{f}$-equivariant linear maps $T_x\mathbf{f} : T_x\mathbf{X} \to T_y\mathbf{Y}$, $O_x\mathbf{f} : O_x\mathbf{X} \to O_y\mathbf{Y}$. These $G_x\mathbf{f}, T_x\mathbf{f}, T_y\mathbf{f}$ are only unique up to the action of an element of $G_y\mathbf{Y}$, so again we use the Axiom of Choice. We may not have $G_x(\mathbf{g} \circ \mathbf{f}) = G_x\mathbf{g} \circ G_x\mathbf{f}$, etc.
If $\boldsymbol{\eta} : \mathbf{f} \Rightarrow \mathbf{g}$ is a 2-morphism of derived orbifolds, there is a canonical element $G_x\boldsymbol{\eta} \in G_y\mathbf{Y}$ such that

$$G_x\mathbf{g}(\gamma) = (G_x\boldsymbol{\eta})(G_x\mathbf{f}(\gamma))(G_x\boldsymbol{\eta})^{-1},$$

$$T_x\mathbf{g}(t) = G_x\boldsymbol{\eta} \cdot T_x\mathbf{f}(t), \quad O_x\mathbf{g}(t) = G_x\boldsymbol{\eta} \cdot O_x\mathbf{f}(t).$$

You can mostly ignore this issue about $G_x\mathbf{X}$, $T_x\mathbf{X}$, $O_x\mathbf{X}$ only being unique up to conjugation by an element of $G_x\mathbf{X}$, it is not very important in practice.

What *is* important is that $T_x\mathbf{X}$, $O_x\mathbf{X}$ are representations of the orbifold group $G_x\mathbf{X}$, so we can think about them using representation theory. For example, we have natural splittings

$$T_x\mathbf{X} = (T_x\mathbf{X})^{\mathrm{tr}} \oplus (T_x\mathbf{X})^{\mathrm{nt}}, \quad O_x\mathbf{X} = (O_x\mathbf{X})^{\mathrm{tr}} \oplus (O_x\mathbf{X})^{\mathrm{nt}}$$

into trivial $(\cdots)^{\mathrm{tr}}$ and nontrivial $(\cdots)^{\mathrm{nt}}$ subrepresentations.

Then it is easy to prove:

Lemma 9.1

Let $\mathbf{X}$ be a derived orbifold and $x \in \mathbf{X}$, and suppose $(O_x\mathbf{X})^{\mathrm{tr}} = 0$ and $(O_x\mathbf{X})^{\mathrm{nt}}$ is not isomorphic to a $G_x\mathbf{X}$-subrepresentation of $(T_x\mathbf{X})^{\mathrm{nt}}$. Then it is not possible to make a small deformation of $\mathbf{X}$ near x so that it becomes a classical orbifold.

In contrast, derived manifolds can always be perturbed to manifolds.

9.2. Immersions, embeddings and d-submanifolds

A smooth map of manifolds $f : X \to Y$ is an *immersion* if $T_xf : T_xX \to T_yY$ is injective for all $x \in X$. It is an *embedding* if also f is a homeomorphism with its image $f(X)$.

Definition

Let $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ be a 1-morphism of derived manifolds. We call $\mathbf{f}$ a *weak immersion*, or *w-immersion*, if $T_x\mathbf{f} : T_x\mathbf{X} \to T_{\mathbf{f}(x)}\mathbf{Y}$ is injective for all $x \in \mathbf{X}$.

We call $\mathbf{f}$ an *immersion* if $T_x\mathbf{f} : T_x\mathbf{X} \to T_{\mathbf{f}(x)}\mathbf{Y}$ is injective and $O_x\mathbf{f} : O_x\mathbf{X} \to O_{\mathbf{f}(x)}\mathbf{Y}$ is surjective for all $x \in \mathbf{X}$.

We call $\mathbf{f}$ a (*w-*)*embedding* if it is a (w-)immersion and $f : X \to f(X)$ is a homeomorphism.

If instead $\mathbf{X}, \mathbf{Y}$ are derived orbifolds, we also require that $G_x\mathbf{f} : G_x\mathbf{X} \to G_{\mathbf{f}(x)}\mathbf{Y}$ is injective for all $x \in \mathbf{X}$ for (w-)immersions, and $G_x\mathbf{f} : G_x\mathbf{X} \to G_{\mathbf{f}(x)}\mathbf{Y}$ is an isomorphism for (w-)embeddings.

Proposition 9.2

Let $\mathbf{S}_{V',f,\hat f}:\mathbf{S}_{V,E,s}\to\mathbf{S}_{W,F,t}$ be a standard model 1-morphism in **dMan***. Then $\mathbf{S}_{V',f,\hat f}$ is a w-immersion (or an immersion) if and only if for all $x\in s^{-1}(0)\subseteq V$ with $f(x)=y\in t^{-1}(0)\subseteq W$, the following sequence is exact at the second term (or the second and fourth terms, respectively):*

$$0\longrightarrow T_xV\xrightarrow{\mathrm{d}s|_x\oplus T_xf}E|_x\oplus T_yW\xrightarrow{\hat f|_x\oplus -\mathrm{d}t|_y}F|_y\longrightarrow 0.$$

Proof.

This follows from the diagram with exact rows, (5.4) in §5:

$$\begin{tikzcd}
0 \arrow[r] & T_x\mathbf{S}_{V,E,s} \arrow[r] \arrow[d, "T_x\mathbf{S}_{V',f,\hat f}"] & T_xV \arrow[r, "\mathrm{d}s|_x"] \arrow[d, "T_xf"] & E|_x \arrow[r] \arrow[d, "\hat f|_x"] & O_x\mathbf{S}_{V,E,s} \arrow[r] \arrow[d, "O_x\mathbf{S}_{V',f,\hat f}"] & 0 \\
0 \arrow[r] & T_y\mathbf{S}_{W,F,t} \arrow[r] & T_yW \arrow[r, "\mathrm{d}t|_y"] & F|_y \arrow[r] & O_y\mathbf{S}_{W,F,t} \arrow[r] & 0.
\end{tikzcd}$$

□

Local models for (w-)immersions

Locally, (w-)immersions are modelled on standard model $\mathbf{S}_{V,f,\hat f}:\mathbf{S}_{V,E,s}\to\mathbf{S}_{W,F,t}$ with $f:V\to W$ an immersion. We can also take $\hat f:E\to f^*(F)$ to be injective/an isomorphism.

Theorem 9.3

Suppose $\mathbf{f}:\mathbf{X}\to\mathbf{Y}$ is a (w-)immersion of derived manifolds, and $x\in\mathbf{X}$ with $\mathbf{f}(x)=y\in\mathbf{Y}$. Then there exists a standard model 1-morphism $\mathbf{S}_{V,f,\hat f}:\mathbf{S}_{V,E,s}\to\mathbf{S}_{W,F,t}$ in a 2-commutative diagram

$$\begin{tikzcd}
\mathbf{S}_{V,E,s} \arrow[rr, "\mathbf{S}_{V,f,\hat f}"] \arrow[d, "\mathbf{i}"] & \arrow[d, phantom, "\Downarrow"] & \mathbf{S}_{W,F,t} \arrow[d, "\mathbf{j}"'] \\
\mathbf{X} \arrow[rr, "\mathbf{f}"] & & \mathbf{Y},
\end{tikzcd}$$

where $\mathbf{i},\mathbf{j}$ are equivalences with open $x\in\mathbf{U}\subseteq\mathbf{X}$, $y\in\mathbf{V}\subseteq\mathbf{Y}$, and $f:V\to W$ is an immersion, and $\hat f:E\to f^*(F)$ is injective (or an isomorphism) if $\mathbf{f}$ is a w-immersion (or an immersion).

Derived submanifolds

An (immersed or embedded) submanifold $X \hookrightarrow Y$ is just an immersion or embedding $i : X \to Y$. For embedded submanifolds we can identify X with its image $i(X) \subseteq Y$, and regard X as a special subset of Y.

To define *derived submanifolds*, we just say that a (w-)immersion or (w-)embedding $\mathbf{i} : \mathbf{X} \to \mathbf{Y}$ is a (w-)immersed or (w-)embedded derived submanifold of $\mathbf{Y}$. We cannot identify $\mathbf{X}$ with a subset of $\mathbf{Y}$ in the (w-)embedded case, though we can think of it as a derived C^∞-subscheme.

We can regard an embedded submanifold $X \subset Y$ as either (i) the image of an embedding $i : X \to Y$, or (ii) locally the solutions of $g_1 = \cdots = g_n = 0$ on Y, for $g_j : Y \to \mathbb{R}$ smooth and transverse.

For an immersion or embedding $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$, we can also write $\mathbf{X}$ locally as the zeroes of $\mathbf{g} : \mathbf{Y} \to \mathbb{R}^n$, but with no transversality.

Theorem 9.4

Suppose $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ is an immersion of derived manifolds, and $x \in \mathbf{X}$ with $\mathbf{f}(x) = y \in \mathbf{Y}$. Then there exist open $x \in \mathbf{U} \subseteq \mathbf{X}$, $y \in \mathbf{V} \subseteq \mathbf{Y}$ with $\mathbf{f}(\mathbf{U}) \subseteq \mathbf{V} \subseteq \mathbf{Y}$, and a 1-morphism $\mathbf{g} : \mathbf{V} \to \mathbb{R}^n$ for $n = \mathrm{vdim}\,\mathbf{Y} - \mathrm{vdim}\,\mathbf{X} \geqslant 0$, in a 2-Cartesian diagram:

$$\begin{tikzcd} \mathbf{U} \arrow[rr] \arrow[d, "\mathbf{f}|_{\mathbf{U}}"] \arrow[drr, phantom, "\Downarrow"] & & * \arrow[d, "0"'] \\ \mathbf{V} \arrow[rr, "\mathbf{g}"] & & \mathbb{R}^n. \end{tikzcd}$$

If $\mathbf{f}$ is an embedding we can take $\mathbf{U} = \mathbf{f}^{-1}(\mathbf{V})$.

Here $\mathbf{U}$ is a *fibre product* $\mathbf{V} \times_{\mathbf{g},\mathbb{R}^n,0} *$, of which more in §10.

9.3. Embedding derived manifolds into manifolds

Theorem 9.5

Suppose $\mathbf{f}:\mathbf{X}\to Y$ is an embedding of d-manifolds, with Y an ordinary manifold. Then there is an equivalence $\mathbf{X}\simeq\mathbf{S}_{V,E,s}$ in **dMan**, *where V is an open neighbourhood of $f(X)$ in Y, and $E\to V$ a vector bundle, and $s\in C^\infty(E)$ with $s^{-1}(0)=f(X)$.*

Sketch proof. (First version was due to David Spivak).

As $\mathbf{f}$ is an embedding, the C^∞-scheme morphism $\underline{f}:\underline{X}\to Y$ is an embedding, so that $\underline{X}$ is a C^∞-subscheme of Y. The relative cotangent complex $\mathbb{L}_{\mathbf{X}/Y}$ is a vector bundle $\underline{E}^*\to\underline{X}$ in degree -1. Take the dual and extend to a vector bundle $E\to V$ on an open neighbourhood V of $f(X)$ in Y. Then we show there exists $s\in C^\infty(E)$ defined near $f(X)=s^{-1}(0)$ such that $\mathbf{X}\simeq V\times_{0,E,s}V$. □

Whitney-style Embedding Theorems

Theorem 9.6 (Whitney 1936)

Let X be a manifold with $\dim X=m$. Then generic smooth maps $f:X\to\mathbb{R}^n$ are immersions if $n\geqslant 2m$, and embeddings if $n\geqslant 2m+1$.

If $\mathbf{X}$ is a derived manifold then 1-morphisms $\mathbf{f}:\mathbf{X}\to\mathbb{R}^n$ form a vector space, and we can take $\mathbf{f}$ to be generic in this.

Theorem 9.7

Let $\mathbf{X}$ be a derived manifold. Then generic 1-morphisms $\mathbf{f}:\mathbf{X}\to\mathbb{R}^n$ are immersions if $n\geqslant 2\dim T_x\mathbf{X}$ for all $x\in\mathbf{X}$, and embeddings if $n\geqslant 2\dim T_x\mathbf{X}+1$ for all $x\in\mathbf{X}$.

Sketch proof.

Near $x\in\mathbf{X}$ we can write $\mathbf{X}\simeq\mathbf{S}_{V,E,s}$ with $\dim V=\dim T_x\mathbf{X}$. Then generic $\mathbf{f}:\mathbf{X}\to\mathbb{R}^n$ factors through generic $g:V\to\mathbb{R}^n$. Apply Theorem 9.6 to see g is an immersion/embedding. □

Combining Theorems 9.5 and 9.7 yields:

Corollary 9.8

A d-manifold $\mathbf{X}$ *is equivalent in* **dMan** *to a standard model d-manifold* $\mathbf{S}_{V,E,s}$ *if and only if* $\dim T_x\mathbf{X}$ *is bounded above for all* $x \in \mathbf{X}$*. This always holds if* $\mathbf{X}$ *is compact.*

Proof.

If the $\dim T_x\mathbf{X}$ are bounded above we can choose $n \geqslant 0$ with $n \geqslant 2\dim T_x\mathbf{X} + 1$ for all $x \in \mathbf{X}$. Then a generic $\mathbf{f} : \mathbf{X} \to \mathbb{R}^n$ is an embedding, and $\mathbf{X} \simeq \mathbf{S}_{V,E,s}$ for V an open neighbourhood of $f(X)$ in $\mathbb{R}^n$. Conversely, if $\mathbf{X} \simeq \mathbf{S}_{V,E,s}$ then $\dim T_x\mathbf{X} \leqslant \dim V$ for $x \in X$. If $\mathbf{X}$ is compact then as $x \mapsto \dim T_x\mathbf{X}$ is upper semicontinuous, $\dim T_x\mathbf{X}$ is bounded above. □

This means that most interesting d-manifolds are principal d-manifolds (i.e. equivalent in **dMan** to some $\mathbf{S}_{V,E,s}$).

For d-orbifolds $\mathbf{X}$, as for Theorem 9.5 we can prove that if $\mathbf{f} : \mathbf{X} \to \mathfrak{Y}$ is an embedding for $\mathfrak{Y}$ an orbifold, then there is an equivalence $\mathbf{X} \simeq \mathfrak{V} \times_{0,\mathfrak{E},s} \mathfrak{V}$ in **dOrb**, where $\mathfrak{V} \subseteq \mathfrak{Y}$ is an open neighbourhood of $\mathbf{f}(\mathbf{X})$ in $\mathfrak{Y}$, and $\mathfrak{E} \to \mathfrak{V}$ an orbifold vector bundle with $s \in C^\infty(\mathfrak{E})$. This uses the condition on embeddings that $G_x\mathbf{f} : G_x\mathbf{X} \to G_{\mathbf{f}(x)}\mathfrak{Y}$ is an isomorphism for all $x \in \mathbf{X}$.
However, we have no good orbifold analogues of Theorem 9.7 or Corollary 9.8. If $\mathbf{X}$ has nontrivial orbifold groups $G_x\mathbf{X} \neq \{1\}$ it cannot have embeddings $\mathbf{f} : \mathbf{X} \to \mathbb{R}^n$, as $G_x\mathbf{f} : G_x\mathbf{X} \to G_{\mathbf{f}(x)}\mathbb{R}^n = \{1\}$ is not an isomorphism.
So we do not have useful criteria for when a d-orbifold can be covered by a single chart $(\mathfrak{V}, \mathfrak{E}, s)$ or (V, E, Γ, s, ψ).

9.4. Submersions

A smooth map of manifolds $f : X \to Y$ is an *submersion* if $T_x f : T_x X \to T_y Y$ is surjective for all $x \in X$. As for (w-)immersions, we have two derived analogues:

Definition

Let $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ be a 1-morphism of derived manifolds or derived orbifolds. We call $\mathbf{f}$ a *weak submersion*, or *w-submersion*, if $O_x\mathbf{f} : O_x\mathbf{X} \to O_{\mathbf{f}(x)}\mathbf{Y}$ is surjective for all $x \in \mathbf{X}$.
We call $\mathbf{f}$ a *submersion* if $T_x\mathbf{f} : T_x\mathbf{X} \to T_{\mathbf{f}(x)}\mathbf{Y}$ is surjective and $O_x\mathbf{f} : O_x\mathbf{X} \to O_{\mathbf{f}(x)}\mathbf{Y}$ is an isomorphism for all $x \in \mathbf{X}$.

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Orientations

Here is the analogue of Proposition 9.2, with the same proof:

Proposition 9.9

Let $\mathbf{S}_{V',f,\hat f} : \mathbf{S}_{V,E,s} \to \mathbf{S}_{W,F,t}$ be a standard model 1-morphism in **dMan**. *Then $\mathbf{S}_{V',f,\hat f}$ is a w-submersion (or a submersion) if and only if for all $x \in s^{-1}(0) \subseteq V$ with $f(x) = y \in t^{-1}(0) \subseteq W$, the following sequence is exact at the fourth term (or the third and fourth terms, respectively):*

$$0 \longrightarrow T_x V \xrightarrow{\mathrm{d}s|_x \oplus T_x f} E|_x \oplus T_y W \xrightarrow{\hat f|_x \oplus -\mathrm{d}t|_y} F|_y \longrightarrow 0.$$

Local models for (w-)submersions

Here is the analogue of Theorem 9.3:

Theorem 9.10

Suppose $\mathbf{f}:\mathbf{X}\to\mathbf{Y}$ is a (w-)submersion of derived manifolds, and $x\in\mathbf{X}$ with $\mathbf{f}(x)=y\in\mathbf{Y}$. Then there exists a standard model 1-morphism $\mathbf{S}_{V,f,\hat f}:\mathbf{S}_{V,E,s}\to\mathbf{S}_{W,F,t}$ in a 2-commutative diagram

$$\begin{tikzcd} \mathbf{S}_{V,E,s} \arrow[rr, "\mathbf{S}_{V,f,\hat f}"] \arrow[d, "\mathbf{i}"] \arrow[drr, phantom, "\Downarrow"] & & \mathbf{S}_{W,F,t} \arrow[d, "\mathbf{j}"'] \\ \mathbf{X} \arrow[rr, "\mathbf{f}"] & & \mathbf{Y}, \end{tikzcd}$$

where $\mathbf{i},\mathbf{j}$ are equivalences with open $x\in\mathbf{U}\subseteq\mathbf{X}$, $y\in\mathbf{V}\subseteq\mathbf{Y}$, and $f:V\to W$ is a submersion, and $\hat f:E\to f^*(F)$ is surjective (or an isomorphism) if $\mathbf{f}$ is a w-submersion (or a submersion).

If $g:X\to Z$ is a submersion of manifolds, then for any smooth map $h:Y\to Z$ the (transverse) fibre product $X\times_{g,Z,h}Y$ exists in **Man**. Here are two derived analogues, explained in §10:

Theorem 9.11

Suppose $\mathbf{g}:\mathbf{X}\to\mathbf{Z}$ is a w-submersion in **dMan**. *Then for any 1-morphism $\mathbf{h}:\mathbf{Y}\to\mathbf{Z}$ in* **dMan**, *the fibre product $\mathbf{X}\times_{\mathbf{g},\mathbf{Z},\mathbf{h}}\mathbf{Y}$ exists in* **dMan**.

Theorem 9.12

Suppose $\mathbf{g}:\mathbf{X}\to\mathbf{Z}$ is a submersion in **dMan**. *Then for any 1-morphism $\mathbf{h}:Y\to\mathbf{Z}$ in* **dMan** *with Y a manifold, the fibre product $\mathbf{X}\times_{\mathbf{g},\mathbf{Z},\mathbf{h}}Y$ exists in* **dMan** *and is a manifold.*
In particular, the fibres $\mathbf{X}_z=\mathbf{X}\times_{\mathbf{g},\mathbf{Z},z}*$ of $\mathbf{g}$ for $z\in\mathbf{Z}$ are manifolds.

Submersions as local projections

If $f : X \to Y$ is a submersion of manifolds and $x \in X$, we can find open $x \in U \subseteq X$ and $f(x) \in V \subseteq Y$ with $f(U) \subseteq V$ and a diffeomorphism $U \cong V \times W$ for some manifold W which identifies $f|_U : U \to V$ with the projection $\pi_V : V \times W \to V$. Here is a derived analogue, which can be deduced from Theorem 9.10:

Theorem 9.13

Let $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ *be a submersion of d-manifolds, and* $x \in \mathbf{X}$ *with* $\mathbf{f}(x) = y \in \mathbf{Y}$*. Then there exist open* $x \in \mathbf{U} \subseteq \mathbf{X}$, $y \in \mathbf{V} \subseteq \mathbf{Y}$ *with* $\mathbf{f}(\mathbf{U}) \subseteq \mathbf{V} \subseteq \mathbf{Y}$*, and an equivalence* $\mathbf{U} \simeq \mathbf{V} \times W$ *for* W *a manifold with* $\dim W = \mathrm{vdim}\, \mathbf{X} - \mathrm{vdim}\, \mathbf{Y} \geqslant 0$*, in a 2-commutative diagram*

$$\begin{tikzcd}
\mathbf{U} \arrow[rr, "\simeq"'] \arrow[drr, bend right, "\mathbf{f}|_{\mathbf{U}}"'] \arrow[dr, phantom, "\Downarrow"] & & \mathbf{V} \times W \arrow[d, "\pi_{\mathbf{V}}"'] \\
& & \mathbf{V}.
\end{tikzcd}$$

9.5. Orientations

Here is one way to define orientations on ordinary manifolds. Let X be a manifold of dimension n. The *canonical bundle* K_X is $\Lambda^n T^*X$. It is a real line bundle over X. An *orientation* o on X is an orientation on the fibres of K_X. That is, o is an equivalence class $[\iota]$ of isomorphisms $\iota : \mathcal{O}_X \to K_X$, where $\mathcal{O}_X = X \times \mathbb{R}$ is the trivial line bundle on X, and two isomorphisms ι, ι' are equivalent if $\iota' = c \cdot \iota$ for $c : X \to (0, \infty)$ a smooth positive function on X. Isomorphisms $\iota : \mathcal{O}_X \to K_X$ are equivalent to non-vanishing n-forms $\omega = \iota(1)$ on X.
The *opposite orientation* is $-o = [-\iota]$.
An *oriented manifold* (X, o) is a manifold X with orientation o. Usually we just say X is an oriented manifold, and write $-X$ for $(X, -o)$ with the opposite orientation.

There is a natural analogue of canonical bundles for derived manifolds and orbifolds.

Theorem 9.14

(a) *Every d-manifold* $\mathbf{X}$ *has a* **canonical bundle** $K_{\mathbf{X}}$, *a* C^∞ *real line bundle over the underlying* C^∞*-scheme* $\underline{X}$, *natural up to canonical isomorphism, with* $K_{\mathbf{X}}|_x \cong \Lambda^{\mathrm{top}} T_x^* \mathbf{X} \otimes \Lambda^{\mathrm{top}} O_x \mathbf{X}$ *for* $x \in \mathbf{X}$.
(b) *If* $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ *is an étale* 1*-morphism (e.g. an equivalence), there is a canonical, functorial isomorphism* $K_{\mathbf{f}} : K_{\mathbf{X}} \to \underline{f}^*(K_{\mathbf{Y}})$. *If* $\mathbf{f}, \mathbf{g} : \mathbf{X} \to \mathbf{Y}$ *are* 2*-isomorphic then* $K_{\mathbf{f}} = K_{\mathbf{g}}$.
(c) *If* $\mathbf{X} \simeq \mathbf{S}_{V,E,s}$, *there is a canonical isomorphism*

$$K_{\mathbf{X}} \cong \left(\Lambda^{\dim V} T^* V \otimes \Lambda^{\mathrm{rank}\, E} E\right)|_{s^{-1}(0)}.$$

Analogues of **(a)–(c)** *hold for d-orbifolds and Kuranishi spaces, with* $K_{\mathbf{X}}$ *an orbifold line bundle over the underlying Deligne–Mumford* C^∞*-stack* $\mathcal{X}$. *In particular, the orbifold groups* $G_x \mathcal{X}$ *can act nontrivially on* $K_{\mathbf{X}}$, *so* $K_{\mathbf{X}}$ *may not be locally trivial.*

To prove Theorem 9.14 for Kuranishi spaces, we show that the line bundles $\left(\Lambda^{\dim V_i} T^* V_i \otimes \Lambda^{\mathrm{rank}\, E_i} E_i\right)|_{s_i^{-1}(0)}$ on $\mathrm{Im}\,\psi_i \subseteq X$ can be glued by canonical isomorphisms on overlaps $\mathrm{Im}\,\psi_i \cap \mathrm{Im}\,\psi_j$.

Definition

An *orientation* o on a d-manifold $\mathbf{X}$ is an equivalence class $[\iota]$ of isomorphisms $\iota : \mathcal{O}_{\underline{X}} \to K_{\mathbf{X}}$, where $\mathcal{O}_{\underline{X}}$ is the trivial line bundle on the C^∞-scheme $\underline{X}$, and two isomorphisms ι, ι' are equivalent if $\iota' = c \cdot \iota$ for $c : \underline{X} \to (0, \infty)$ a smooth positive function on $\underline{X}$.
An *oriented d-manifold* $(\mathbf{X}, o)$ is a d-manifold $\mathbf{X}$ with orientation o. Usually we just say $\mathbf{X}$ is an oriented d-manifold, and write $-\mathbf{X}$ for $(\mathbf{X}, -o)$ with the opposite orientation.
We make similar definitions for d-orbifolds and Kuranishi spaces.

An orientation on $\mathbf{S}_{V,E,s}$ is equivalent to an orientation (near $s^{-1}(0)$) on the total space of E.

Derived Differential Geometry

Lecture 10 of 14: Fibre products of derived manifolds and orbifolds

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Summer 2015

These slides, and references, etc., available at
http://people.maths.ox.ac.uk/∼joyce/DDG2015

Plan of talk:

10. Fibre products of derived manifolds and orbifolds

10.1. D-transverse fibre products of derived manifolds

Fibre products of derived manifolds and orbifolds are very important. Standard models $\mathbf{S}_{V,E,s}$ are fibre products $V \times_{0,E,s} V$ in **dMan**. Theorems 9.4, 9.11, and 9.12 in §9 involved fibre products. Applications often involve fibre products, for example moduli spaces $\bar{\mathcal{M}}_k(\beta)$ of prestable J-holomorphic discs Σ in a symplectic manifold (M,ω) with boundary in a Lagrangian L, relative homology class $[\Sigma]=\beta\in H_2(M,L;\mathbb{Z})$, and k boundary marked points, are Kuranishi spaces with corners satisfying

$$\partial\bar{\mathcal{M}}_k(\beta)=\coprod_{i+j=k}\coprod_{\beta_1+\beta_2=\beta}\bar{\mathcal{M}}_{i+1}(\beta_1)\times_{\mathbf{ev}_{i+1},L,\mathbf{ev}_{j+1}}\bar{\mathcal{M}}_{j+1}(\beta_2). \quad (10.1)$$

Recall that smooth maps of manifolds $g:X\to Z$, $h:Y\to Z$ are *transverse* if for all $x\in X$, $y\in Y$ with $g(x)=h(y)=z\in Z$, then $T_xg\oplus T_yh:T_xX\oplus T_yY\to T_zZ$ is surjective. If g,h are transverse then a fibre product $W=X\times_{g,Z,h}Y$ exists in **Man**, with $\dim W=\dim X+\dim Y-\dim Z$.
We give two derived analogues of transversality, weak and strong:

Definition 10.1

Let $\mathbf{g}:\mathbf{X}\to\mathbf{Z}$, $\mathbf{h}:\mathbf{Y}\to\mathbf{Z}$ be 1-morphisms of d-manifolds. We call $\mathbf{g},\mathbf{h}$ *d-transverse* if for all $x\in\mathbf{X}$, $y\in\mathbf{Y}$ with $\mathbf{g}(x)=\mathbf{h}(y)=z$ in $\mathbf{Z}$, then $O_x\mathbf{g}\oplus O_y\mathbf{h}:O_x\mathbf{X}\oplus O_y\mathbf{Y}\to O_z\mathbf{Z}$ is surjective.

We call $\mathbf{g},\mathbf{h}$ *strongly d-transverse* if for all $x\in X$, $y\in Y$ with $g(x)=h(y)=z\in Z$, then $T_x\mathbf{g}\oplus T_y\mathbf{h}:T_x\mathbf{X}\oplus T_y\mathbf{Y}\to T_z\mathbf{Z}$ is surjective, and $O_x\mathbf{g}\oplus O_y\mathbf{h}:O_x\mathbf{X}\oplus O_y\mathbf{Y}\to O_z\mathbf{Z}$ is an isomorphism.

Here is the main result:

Theorem 10.2

Suppose $\mathbf{g}:\mathbf{X}\to\mathbf{Z}$, $\mathbf{h}:\mathbf{Y}\to\mathbf{Z}$ *are d-transverse 1-morphisms in* **dMan**. *Then a fibre product* $\mathbf{W}=\mathbf{X}\times_{\mathbf{g},\mathbf{Z},\mathbf{h}}\mathbf{Y}$ *exists in the 2-category* **dMan**, *with* $\mathrm{vdim}\,\mathbf{W}=\mathrm{vdim}\,\mathbf{X}+\mathrm{vdim}\,\mathbf{Y}-\mathrm{vdim}\,\mathbf{Z}$. *This* $\mathbf{W}$ *is a manifold if and only if* $\mathbf{g},\mathbf{h}$ *are strongly d-transverse. The topological space* W *of* $\mathbf{W}$ *is given by*

$$W=\bigl\{(x,y)\in X\times Y:\mathbf{g}(x)=\mathbf{h}(y)\ \text{in}\ Z\bigr\}. \tag{10.2}$$

For all $(x,y)\in W$ *with* $\mathbf{g}(x)=\mathbf{h}(y)=z$ *in* $\mathbf{Z}$, *there is a natural long exact sequence*

$$\begin{tikzcd}[column sep=large]
0 \arrow[r] & T_{(x,y)}\mathbf{W} \arrow[r, "T_{(x,y)}\mathbf{e}\oplus -T_{(x,y)}\mathbf{f}"'] & T_x\mathbf{X}\oplus T_y\mathbf{Y} \arrow[r, "T_x\mathbf{g}\oplus T_y\mathbf{h}"'] & T_z\mathbf{Z} \arrow[d] \\
0 & O_z\mathbf{Z} \arrow[l] & O_x\mathbf{X}\oplus O_y\mathbf{Y} \arrow[l, "O_x\mathbf{g}\oplus O_y\mathbf{h}"'] & O_{(x,y)}\mathbf{W}, \arrow[l, "O_{(x,y)}\mathbf{e}\oplus -O_{(x,y)}\mathbf{f}"']
\end{tikzcd} \tag{10.3}$$

where $\mathbf{e}:\mathbf{W}\to\mathbf{X}$, $\mathbf{f}:\mathbf{W}\to\mathbf{Y}$ *are the projections.*

Note that quite a lot of the theorem can be seen from the exact sequence (10.3). D-transversality of $\mathbf{g},\mathbf{h}$ is equivalent to exactness at $O_z\mathbf{Z}$. The rest of the sequence determines $T_{(x,y)}\mathbf{W}$, $O_{(x,y)}\mathbf{W}$. In particular, $O_{(x,y)}\mathbf{W}$ is the direct sum of the cokernel of $T_x\mathbf{g}\oplus T_y\mathbf{h}:T_x\mathbf{X}\oplus T_y\mathbf{Y}\to T_z\mathbf{Z}$ and the kernel of $O_x\mathbf{g}\oplus O_y\mathbf{h}:O_x\mathbf{X}\oplus O_y\mathbf{Y}\to O_z\mathbf{Z}$. Therefore $O_{(x,y)}\mathbf{W}=0$ for all $(x,y)\in\mathbf{W}$ if and only if $\mathbf{g},\mathbf{h}$ are strongly d-transverse. But a d-manifold $\mathbf{W}$ is a manifold if and only if $O_w\mathbf{W}=0$ for all $w\in\mathbf{W}$. The equation $\mathrm{vdim}\,\mathbf{W}=\mathrm{vdim}\,\mathbf{X}+\mathrm{vdim}\,\mathbf{Y}-\mathrm{vdim}\,\mathbf{Z}$ holds by taking alternating sums of dimensions in (10.3) and using $\mathrm{vdim}\,\mathbf{X}=\dim T_x\mathbf{X}-\dim O_x\mathbf{X}$, etc.

Here is a way to think about d-transversality. Work in a suitable ∞-category $\mathbf{DerC^\infty Sch}$ of derived C^∞-schemes, such as that defined by Spivak. Then derived manifolds $\mathbf{X}$ are objects in $\mathbf{DerC^\infty Sch}$, which are *quasi-smooth*, that is, the cotangent complex $\mathbb{L}_{\mathbf{X}}$ lives in degrees $[-1,0]$. If $\mathbf{X}$ is a manifold (it is *smooth*), its cotangent complex lives in degree 0 only.

For any 1-morphisms $\mathbf{g}:\mathbf{X}\to\mathbf{Z}$, $\mathbf{h}:\mathbf{Y}\to\mathbf{Z}$ of derived manifolds, a fibre product $\mathbf{W}=\mathbf{X}\times_{\mathbf{g},\mathbf{Z},\mathbf{h}}\mathbf{Y}$ exists in $\mathbf{DerC^\infty Sch}$ (as all fibre products do), with topological space W as in (10.2). The cotangent complexes form a distinguished triangle of $\mathcal{O}^\bullet_{\mathbf{W}}$-modules:

$$\cdots \longrightarrow (\mathbf{g}\circ\mathbf{e})^*(\mathbb{L}_{\mathbf{Z}}) \longrightarrow \mathbf{e}^*(\mathbb{L}_{\mathbf{X}})\oplus\mathbf{f}^*(\mathbb{L}_{\mathbf{Y}}) \longrightarrow \mathbb{L}_{\mathbf{W}} \xrightarrow{[1]} \cdots \qquad (10.4)$$

Therefore $\mathbb{L}_{\mathbf{W}}$ lives in degrees $[-2,0]$, so in general $\mathbf{W}$ is not quasi-smooth, and not a derived manifold. D-transversality is the necessary and sufficient condition for $H^{-2}(\mathbb{L}_{\mathbf{W}}|_{(x,y)})=0$ for all $(x,y)\in\mathbf{W}$, so that $\mathbb{L}_{\mathbf{W}}$ lives in degrees $[-1,0]$. Then (10.3) is the dual of the cohomology exact sequence of (10.4) restricted to (x,y).

Fibre products over manifolds

If $\mathbf{Z}$ is a manifold then $O_z\mathbf{Z}=0$ for all $z\in\mathbf{Z}$, so any $\mathbf{g}:\mathbf{X}\to\mathbf{Z}$, $\mathbf{h}:\mathbf{Y}\to\mathbf{Z}$ are d-transverse. Thus Theorem 10.2 gives:

Corollary 10.3

Suppose $\mathbf{g}:\mathbf{X}\to\mathbf{Z}$, $\mathbf{h}:\mathbf{Y}\to\mathbf{Z}$ *are* 1*-morphisms in* $\mathbf{dMan}$, *with* $\mathbf{Z}$ *a manifold. Then a fibre product* $\mathbf{W}=\mathbf{X}\times_{\mathbf{g},\mathbf{Z},\mathbf{h}}\mathbf{Y}$ *exists in* $\mathbf{dMan}$, *with* $\mathrm{vdim}\,\mathbf{W}=\mathrm{vdim}\,\mathbf{X}+\mathrm{vdim}\,\mathbf{Y}-\dim\mathbf{Z}$.

This is very useful. For example, the symplectic geometry equation (10.1) involves fibre products over a manifold (in the d-orbifold case), which automatically exist.

W-submersions and submersions

Recall from §9.4 that $\mathbf{g}:\mathbf{X}\to\mathbf{Z}$ in $\mathbf{dMan}$ is a *w-submersion* if $O_x\mathbf{g}:O_x\mathbf{X}\to O_z\mathbf{Z}$ is surjective for all $x\in\mathbf{X}$. For any other 1-morphism $\mathbf{h}:\mathbf{Y}\to\mathbf{Z}$ in $\mathbf{dMan}$, this implies that $O_x\mathbf{g}\oplus O_y\mathbf{h}:O_x\mathbf{X}\oplus O_y\mathbf{Y}\to O_z\mathbf{Z}$ is surjective, so $\mathbf{g},\mathbf{h}$ are d-transverse, and thus a fibre product $\mathbf{W}=\mathbf{X}\times_{\mathbf{g},\mathbf{Z},\mathbf{h}}\mathbf{Y}$ exists in $\mathbf{dMan}$ by Theorem 10.2. This proves Theorem 9.11.

Also $\mathbf{g}:\mathbf{X}\to\mathbf{Z}$ is a *submersion* if $T_x\mathbf{g}:T_x\mathbf{X}\to T_z\mathbf{Z}$ is surjective and $O_x\mathbf{g}:O_x\mathbf{X}\to O_z\mathbf{Z}$ is an isomorphism for all $x\in\mathbf{X}$. For any other 1-morphism $\mathbf{h}:\mathbf{Y}\to\mathbf{Z}$ in $\mathbf{dMan}$ with $\mathbf{Y}$ a manifold, this implies that $T_x\mathbf{g}\oplus T_y\mathbf{h}:T_x\mathbf{X}\oplus T_y\mathbf{Y}\to T_z\mathbf{Z}$ is surjective, and $O_x\mathbf{g}\oplus O_y\mathbf{h}:O_x\mathbf{X}\oplus O_y\mathbf{Y}\to O_z\mathbf{Z}$ is an isomorphism, as $O_y\mathbf{Y}=0$. So $\mathbf{g},\mathbf{h}$ are strongly d-transverse, and $\mathbf{W}=\mathbf{X}\times_{\mathbf{g},\mathbf{Z},\mathbf{h}}\mathbf{Y}$ exists and is a manifold by Theorem 10.2. This proves Theorem 9.12.

Non d-transverse fibre products

If $\mathbf{g}:\mathbf{X}\to\mathbf{Z}$ and $\mathbf{h}:\mathbf{Y}\to\mathbf{Z}$ are 1-morphisms in $\mathbf{dMan}$ which are not d-transverse, then a fibre product $\mathbf{W}=\mathbf{X}\times_{\mathbf{g},\mathbf{Z},\mathbf{h}}\mathbf{Y}$ may or may not exist in $\mathbf{dMan}$. If it does exist, then generally we have $\mathrm{vdim}\,\mathbf{W}<\mathrm{vdim}\,\mathbf{X}+\mathrm{vdim}\,\mathbf{Y}-\mathrm{vdim}\,\mathbf{Z}$.

Example 10.4

Let $\mathbf{X}=\mathbf{Y}=*$ be a point, and $\mathbf{Z}=\mathbf{S}_{*,\mathbb{R}^n,0}$ be the standard model d-manifold which is a point $*$ with obstruction space $O_*\mathbf{Z}=\mathbb{R}^n$ for $n>0$. Let $\mathbf{g}:\mathbf{X}\to\mathbf{Z}$, $\mathbf{h}:\mathbf{Y}\to\mathbf{Z}$ be the unique 1-morphisms, the standard model 1-morphism $\mathbf{S}_{*,\mathrm{id}_*,0}$.
Then $O_*\mathbf{g}$, $O_*\mathbf{h}$ map $0\to\mathbb{R}^n$, so $\mathbf{g},\mathbf{h}$ are not d-transverse. The fibre product $\mathbf{W}=\mathbf{X}\times_{\mathbf{g},\mathbf{Z},\mathbf{h}}\mathbf{Y}$ exists in $\mathbf{dMan}$ and is the point $*$. So $\mathrm{vdim}\,\mathbf{W}=0<\mathrm{vdim}\,\mathbf{X}+\mathrm{vdim}\,\mathbf{Y}-\mathrm{vdim}\,\mathbf{Z}=0+0-(-n)=n$.

10.2. Fibre products of derived orbifolds

For fibre products in the 2-categories of d-orbifolds **dOrb** or (equivalently) Kuranishi spaces **Kur**, an analogue of Theorem 10.2 holds. We have to be careful of two points:

(a) As in §9.1, if $\mathbf{g}:\mathbf{X}\to\mathbf{Z}$ is a 1-morphism in **dOrb** or **Kur** and $x\in\mathbf{X}$ with $\mathbf{g}(x)=z\in\mathbf{Z}$, then $T_x\mathbf{g}:T_x\mathbf{X}\to T_z\mathbf{Z}$ and $O_x\mathbf{g}:O_x\mathbf{X}\to O_z\mathbf{Z}$ are only naturally defined up to the action of an element of the orbifold group $G_z\mathbf{Z}$ on $T_z\mathbf{Z}, O_z\mathbf{Z}$. We must include this in the definition of (strong) d-transversality.

(b) When a d-transverse fibre product $\mathbf{W}=\mathbf{X}\times_{\mathbf{g},\mathbf{Z},\mathbf{h}}\mathbf{Y}$ exists, the underlying topological space is generally not $W=\{(x,y)\in X\times Y:\mathbf{g}(x)=\mathbf{h}(y)\}$, as in (10.2). Instead, the continuous map $W\to X\times Y$ is finite, but not injective, as the fibre over $(x,y)\in X\times Y$ with $\mathbf{g}(x)=\mathbf{h}(y)=z\in\mathbf{Z}$ is $G_x\mathbf{g}(G_x\mathbf{X})\backslash G_z\mathbf{Z}/G_y\mathbf{h}(G_y\mathbf{Y})$.

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For (a), here is the orbifold analogue of Definition 10.1:

Definition 10.5

Let $\mathbf{g}:\mathbf{X}\to\mathbf{Z}$, $\mathbf{h}:\mathbf{Y}\to\mathbf{Z}$ be 1-morphisms of derived orbifolds. We call $\mathbf{g},\mathbf{h}$ *d-transverse* if for all $x\in\mathbf{X}$, $y\in\mathbf{Y}$ with $\mathbf{g}(x)=\mathbf{h}(y)=z$ in $\mathbf{Z}$, and all $\gamma\in G_z\mathbf{Z}$, then $O_x\mathbf{g}\oplus(\gamma\cdot O_y\mathbf{h}):O_x\mathbf{X}\oplus O_y\mathbf{Y}\to O_z\mathbf{Z}$ is surjective.

We call $\mathbf{g},\mathbf{h}$ *strongly d-transverse* if for all $x\in X$, $y\in Y$ with $g(x)=h(y)=z\in Z$, and all $\gamma\in G_z\mathbf{Z}$, then $T_x\mathbf{g}\oplus(\gamma\cdot T_y\mathbf{h}):T_x\mathbf{X}\oplus T_y\mathbf{Y}\to T_z\mathbf{Z}$ is surjective, and $O_x\mathbf{g}\oplus(\gamma\cdot O_y\mathbf{h}):O_x\mathbf{X}\oplus O_y\mathbf{Y}\to O_z\mathbf{Z}$ is an isomorphism.

Here is the orbifold analogue of Theorem 10.2:

Theorem 10.6

Suppose $\mathbf{g}:\mathbf{X}\to\mathbf{Z}$, $\mathbf{h}:\mathbf{Y}\to\mathbf{Z}$ *are d-transverse* 1*-morphisms in* **dOrb** *or* **Kur**. *Then a fibre product* $\mathbf{W}=\mathbf{X}\times_{\mathbf{g},\mathbf{Z},\mathbf{h}}\mathbf{Y}$ *exists in* **dOrb** *or* **Kur**, *with* $\mathrm{vdim}\,\mathbf{W}=\mathrm{vdim}\,\mathbf{X}+\mathrm{vdim}\,\mathbf{Y}-\mathrm{vdim}\,\mathbf{Z}$.
This $\mathbf{W}$ *is an orbifold if and only if* $\mathbf{g},\mathbf{h}$ *are strongly d-transverse.*
The topological space W *of* $\mathbf{W}$ *is given as a set by*

$$\begin{aligned} W=\bigl\{(x,y,C):x\in X,\;\; y\in Y,\;\; \mathbf{g}(x)=\mathbf{h}(y)=z\in Z,\\ C\in G_x\mathbf{g}(G_x\mathbf{X})\backslash G_z\mathbf{Z}/G_y\mathbf{h}(G_y\mathbf{Y})\bigr\}.\end{aligned} \tag{10.5}$$

For $(x,y,C)\in W$ *with* $\gamma\in C\subseteq G_z\mathbf{Z}$, *there is an exact sequence*

$$\begin{array}{ccccccc} 0\to T_{(x,y,C)}\mathbf{W} & \longrightarrow & T_x\mathbf{X}\oplus T_y\mathbf{Y} & \xrightarrow[T_x\mathbf{g}\oplus(\gamma\cdot T_y\mathbf{h})]{} & T_z\mathbf{Z} \\ & & & & \downarrow \\ 0\longleftarrow O_z\mathbf{Z} & \xleftarrow{O_x\mathbf{g}\oplus(\gamma\cdot O_y\mathbf{h})} & O_x\mathbf{X}\oplus O_y\mathbf{Y} & \longleftarrow & O_{(x,y,C)}\mathbf{W}. \end{array} \tag{10.6}$$

Transverse fibre products of quotient orbifolds

Example 10.7

Suppose U,V,W are manifolds, Γ,Δ,K are finite groups acting smoothly on U,V,W, $\rho:\Gamma\to K$, $\sigma:\Delta\to K$ are group morphisms, and $g:U\to W$, $h:V\to W$ are ρ-, σ-equivariant smooth maps. Then $[g,\rho]:[U/\Gamma]\to[W/K]$, $[h,\sigma]:[V/\Delta]\to[W/K]$ are smooth maps of orbifolds.
We say that $[g,\rho],[h,\sigma]$ are *transverse* if $g:U\to W$ and $\kappa\cdot h:V\to W$ are transverse maps of manifolds for all $\kappa\in K$.
If $[g,\rho],[h,\sigma]$ are transverse then the fibre product in **Orb** is

$$[U/\Gamma]\times_{[g,\rho],[W/K],[h,\sigma]}[V/\Delta]\simeq\bigl[\bigl(U\times_{g,W,K\cdot h}(V\times K)\bigr)/(\Gamma\times\Delta)\bigr].$$

Here $K\cdot h:V\times K\to W$ maps $(v,\kappa)\mapsto\kappa\cdot h(v)$, and $\Gamma\times\Delta$ acts on $U\times_{g,W,K\cdot h}(V\times K)$ by $(\gamma,\delta):(u,v,\kappa)\mapsto\bigl(\gamma\cdot u,\delta\cdot v,\rho(\gamma)\kappa\sigma(\delta)^{-1}\bigr)$.

In particular, if $U = V = W = *$ then we have

$$[*/\Gamma] \times_{[*/K]} [*/\Delta] = [\Gamma \backslash K/\Delta].$$

This explains the double quotient $G_x \mathbf{g}(G_x \mathbf{X}) \backslash G_z \mathbf{Z}/G_y \mathbf{h}(G_y \mathbf{Y})$ in (10.5). If $\Gamma = \Delta = \{1\}$ then we have

$$* \times_{[*/K]} * = K,$$

considered as a 0-manifold with the discrete topology. So although the topological spaces X, Y, Z are all single points $*$, the topological space W is $|K|$ points, and (10.2) fails if $|K| > 1$.

10.3. Sketch proof of Theorem 10.2

Recall that d-manifolds **dMan** are a full 2-subcategory of d-spaces **dSpa**, a kind of derived C^∞-scheme. We first prove that all fibre products exist in **dSpa**, and that the forgetful functor to topological spaces **dSpa** $\rightarrow$ **Top** preserves fibre products. We do this by writing down an explicit fibre product $\mathbf{W} = \mathbf{X} \times_{\mathbf{g},\mathbf{Z},\mathbf{h}} \mathbf{Y}$ for any 1-morphisms $\mathbf{g} : \mathbf{X} \rightarrow \mathbf{Z}$, $\mathbf{h} : \mathbf{Y} \rightarrow \mathbf{Z}$ in **dSpa**, and verifying it satisfies the universal property.

Thus, if $\mathbf{g} : \mathbf{X} \rightarrow \mathbf{Z}$, $\mathbf{h} : \mathbf{Y} \rightarrow \mathbf{Z}$ are d-transverse 1-morphisms in **dMan**, then a fibre product $\mathbf{W} = \mathbf{X} \times_{\mathbf{g},\mathbf{Z},\mathbf{h}} \mathbf{Y}$ exists in **dSpa**, with topological space

$$W = \{(x, y) \in X \times Y : \mathbf{g}(x) = \mathbf{h}(y) \text{ in } Z\}.$$

If we can show $\mathbf{W}$ is a d-manifold, with $\mathrm{vdim}\,\mathbf{W} = \mathrm{vdim}\,\mathbf{X} + \mathrm{vdim}\,\mathbf{Y} - \mathrm{vdim}\,\mathbf{Z}$, then $\mathbf{W}$ is also a fibre product in **dMan**, as the universal property in **dSpa** implies that in **dMan**.

For $\mathbf{W}$ to be a d-manifold with given dimension is a local property: we have to show an open neighbourhood of each (x,y) in $\mathbf{W}$ is a d-manifold of given dimension.
So let $x\in\mathbf{X}$, $y\in\mathbf{Y}$ with $\mathbf{g}(x)=\mathbf{h}(y)=z\in\mathbf{Z}$. We can choose small open neighbourhoods $x\in\mathbf{T}\subseteq\mathbf{X}$, $y\in\mathbf{U}\subseteq\mathbf{Y}$, $z\in\mathbf{V}\subseteq\mathbf{Z}$ with $\mathbf{g}(\mathbf{T}),\mathbf{h}(\mathbf{U})\subseteq\mathbf{V}$, and equivalences $\mathbf{T}\simeq\mathbf{S}_{T,E,t}$, $\mathbf{U}\simeq\mathbf{S}_{U,F,u}$, $\mathbf{V}\simeq\mathbf{S}_{V,G,v}$ with standard model d-manifolds.
By making $\mathbf{T},\mathbf{U},\mathbf{V}$ smaller we suppose $V\subseteq\mathbb{R}^n$ is open, and $G\to V$ is a trivial vector bundle $V\times\mathbb{R}^k\to V$.
Then as in §5.3, $\mathbf{g}|_{\mathbf{T}}:\mathbf{T}\to\mathbf{V}$, $\mathbf{h}|_{\mathbf{U}}:\mathbf{U}\to\mathbf{V}$ are equivalent to standard model 1-morphisms $\mathbf{S}_{T',g,\hat g}:\mathbf{S}_{T,E,t}\to\mathbf{S}_{V,G,v}$ and $\mathbf{S}_{U',h,\hat h}:\mathbf{S}_{U,F,u}\to\mathbf{S}_{V,G,v}$.
Thus, the open neighbourhood $\mathbf{T}\times_{\mathbf{g}|_{\mathbf{T}},\mathbf{Z},\mathbf{h}|_{\mathbf{U}}}\mathbf{U}$ of (x,y) in $\mathbf{W}$ is equivalent in $\mathbf{dSpa}$ to the fibre product $\mathbf{S}_{T,E,t}\times_{\mathbf{S}_{T',g,\hat g},\mathbf{S}_{V,G,v},\mathbf{S}_{U',h,\hat h}}\mathbf{S}_{U,F,u}$ in $\mathbf{dSpa}$.

Differential Geometry of derived manifolds and orbifolds
Fibre products of derived manifolds and orbifolds
D-transverse fibre products of derived manifolds
Fibre products of derived orbifolds
Sketch proof of Theorem 10.2
Orientations on fibre products

This reduces proving Theorem 10.2 to showing that fibre products $\mathbf{S}_{T,E,t}\times_{\mathbf{S}_{T',g,\hat g},\mathbf{S}_{V,G,v},\mathbf{S}_{U',h,\hat h}}\mathbf{S}_{U,F,u}$ of standard model d-manifolds by d-transverse standard model 1-morphisms exist in $\mathbf{dMan}$, and have the expected dimension, and long exact sequence (10.3), where we make the simplifying assumptions that $V\subseteq\mathbb{R}^n$ is open and $G=V\times\mathbb{R}^k$ is a trivial vector bundle.
We prove this by defining a standard model d-manifold $\mathbf{S}_{S,D,s}$, and showing it is 1-isomorphic in $\mathbf{dSpa}$ to the explicit fibre product $\mathbf{S}_{T,E,t}\times_{\mathbf{S}_{T',g,\hat g},\mathbf{S}_{V,G,v},\mathbf{S}_{U',h,\hat h}}\mathbf{S}_{U,F,u}$ already constructed in $\mathbf{dSpa}$.
Explicitly, we take S to be an open neighbourhood of $\bigl\{(x,y)\in t^{-1}(0)\times u^{-1}(0):g(x)=h(y)\text{ in }v^{-1}(0)\bigr\}$ in $T'\times U'$.
On S we have a morphism of vector bundles

$$\pi_T^*(\hat g)\oplus\pi_U^*(\hat h)\oplus A:\pi_T^*(E)\oplus\pi_U^*(F)\oplus\mathbb{R}^n\longrightarrow\mathbb{R}^k \qquad (10.7)$$

where $A:S\times\mathbb{R}^n\to\mathbb{R}^k$ is constructed from $v:V\to\mathbb{R}^k$ using Hadamard's Lemma. For S small enough, d-transversality implies (10.7) is surjective, and we define $D\to S$ to be its kernel.

10.4. Orientations on fibre products

Suppose X, Y, Z are oriented smooth manifolds, and $g : X \to Z$, $h : Y \to Z$ are transverse smooth maps. Then on the fibre product $W = X \times_{g,Z,h} Y$ in **Man** we can define an orientation, depending on the orientations of X, Y, Z, which is natural, except that it depends on a choice of *orientation convention*. A different orientation convention would multiply the orientation of W by a sign depending on $\dim X, \dim Y, \dim Z$.
We will first explain how to define the orientation on W in the classical case, and then generalize all of this to d-transverse fibre products of derived manifolds and orbifolds.

Recall from §9.5 that if X is an n-manifold, the *canonical bundle* is the real line bundle $K_X = \Lambda^n T^*X$ over X, and an orientation on X is an orientation on the fibres of K_X. To orient fibre products, we first show that given a transverse Cartesian square in **Man**

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "e"] & Y \arrow[d, "h"] \\ X \arrow[r, "g"] & Z, \end{tikzcd} \tag{10.8}$$

there is a natural isomorphism (depending on orientation convention)

$$K_W \cong e^*(K_X) \otimes f^*(K_Y) \otimes (g \circ e)^*(K_Z)^*. \tag{10.9}$$

Thus orientations on the fibres of K_X, K_Y, K_Z determine an orientation on the fibres of K_W, and hence an orientation on W.

Proposition 10.8

Suppose we are given an exact sequence of finite-dimensional $\mathbb{R}$-vector spaces, with $E^i = 0$ for $|i| \gg 0$

$$\cdots \xrightarrow{\mathrm{d}} E^{i-1} \xrightarrow{\mathrm{d}} E^i \xrightarrow{\mathrm{d}} E^{i+1} \xrightarrow{\mathrm{d}} \cdots .$$

Then there is an isomorphism, depending on orientation convention

$$\bigotimes_{i\in\mathbb{Z},\ i\ \text{odd}} \Lambda^{\mathrm{top}} E^i \cong \bigotimes_{i\in\mathbb{Z},\ i\ \text{even}} \Lambda^{\mathrm{top}} E^i .$$

Proof.

Using kernels and cokernels, we can choose vector spaces V^i for $i\in\mathbb{Z}$ and isomorphisms $E^i \cong V^i \oplus V^{i+1}$, such that $\mathrm{d}: E^i \to E^{i+1}$ is

$$\begin{pmatrix} 0 & \mathrm{id}_{V_{i+1}} \\ 0 & 0 \end{pmatrix} : \begin{matrix} V^i \oplus \\ V^{i+1} \end{matrix} \longrightarrow \begin{matrix} V^{i+1} \oplus \\ V^{i+2} \end{matrix}.$$

Then

$$\bigotimes_{i\in\mathbb{Z},\ i\ \text{odd}} \Lambda^{\mathrm{top}} E^i \cong \bigotimes_{i\in\mathbb{Z}} \Lambda^{\mathrm{top}} V^i \cong \bigotimes_{i\in\mathbb{Z},\ i\ \text{even}} \Lambda^{\mathrm{top}} E^i .$$

□

Given a transverse Cartesian square (10.8), to define the isomorphism (10.9), note that we have an exact sequence

$$0 \to \begin{matrix}(g\circ e)^* \\ (T^*Z)\end{matrix} \xrightarrow{e^*(T^*g)\oplus f^*(T^*h)} \begin{matrix} e^*(T^*X) \\ \oplus f^*(T^*Y)\end{matrix} \xrightarrow{T^*e\oplus -T^*f} T^*W \to 0. \quad (10.10)$$

Applying Proposition 10.8 to (10.10) gives an isomorphism

$$K_W \otimes (g\circ e)^*(K_Z) \cong e^*(K_X) \otimes f^*(K_Y),$$

and rearranging gives (10.9). The 'orientation convention' is the choice of where to put signs in (10.10), how to identify $\Lambda^{\mathrm{top}}(U\oplus V) \cong (\Lambda^{\mathrm{top}} U)\otimes_{\mathbb{R}} (\Lambda^{\mathrm{top}} V)$, whether to write $E^i = V^i \oplus V^{i+1}$ or $E^i = V^{i+1}\oplus V^i$ in Proposition 10.8, and so on.

Theorem 10.9

Suppose we are given a d-transverse 2-Cartesian square in **dMan**

$$\begin{tikzcd} \mathbf{W} \arrow[rr, "\mathbf{f}"] \arrow[d, "\mathbf{e}"] & \arrow[d, phantom, "\eta\Uparrow"] & \mathbf{Y} \arrow[d, "\mathbf{h}"] \\ \mathbf{X} \arrow[rr, "\mathbf{g}"] & & \mathbf{Z}. \end{tikzcd}$$

Then there is a canonical isomorphism of C^∞-line bundles on the C^∞-scheme $\underline{W}$, depending on an orientation convention:

$$K_{\mathbf{W}} \cong \underline{e}^*(K_{\mathbf{X}}) \otimes_{\mathcal{O}_W} \underline{f}^*(K_{\mathbf{Y}}) \otimes_{\mathcal{O}_W} (\underline{g} \circ \underline{e})^*(K_{\mathbf{Z}})^*. \tag{10.11}$$

Hence orientations on $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ *induce an orientation on* $\mathbf{W}$.

Sketch proof.

We have $K_{\mathbf{X}}|_x \cong (\Lambda^{\mathrm{top}} T_x\mathbf{X})^* \otimes (\Lambda^{\mathrm{top}} O_x\mathbf{X})$. Applying Proposition 10.8 to the long exact sequence (10.6) gives an isomorphism

$$K_{\mathbf{W}}|_w \cong K_{\mathbf{X}}|_x \otimes K_{\mathbf{Y}}|_y \otimes K_{\mathbf{Z}}^*|_z$$

for all $w \in \mathbf{W}$ with $\mathbf{e}(w) = x \in \mathbf{X}$, $\mathbf{f}(w) = y \in \mathbf{Y}$ and $\mathbf{g}(x) = \mathbf{h}(y) = z \in \mathbf{Z}$. Doing this in families gives (10.11). □

Fibre products have commutativity and associativity properties, up to canonical equivalence. The corresponding orientations given by Theorem 10.9 differ by a sign depending on the dimensions, and the orientation convention. For example, with my orientation conventions, if $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ are oriented d-manifolds and $\mathbf{g} : \mathbf{X} \to \mathbf{Z}$, $\mathbf{h} : \mathbf{Y} \to \mathbf{Z}$ are d-transverse then in oriented d-manifolds we have

$$\mathbf{X} \times_{\mathbf{g},\mathbf{Z},\mathbf{h}} \mathbf{Y} \simeq (-1)^{(\mathrm{vdim}\,\mathbf{X} - \mathrm{vdim}\,\mathbf{Z})(\mathrm{vdim}\,\mathbf{Y} - \mathrm{vdim}\,\mathbf{Z})} \mathbf{Y} \times_{\mathbf{h},\mathbf{Z},\mathbf{g}} \mathbf{X}.$$

When $\mathbf{Z} = *$ so that $\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y} = \mathbf{X} \times \mathbf{Y}$ this gives

$$\mathbf{X} \times \mathbf{Y} \simeq (-1)^{\mathrm{vdim}\,\mathbf{X}\,\mathrm{vdim}\,\mathbf{Y}} \mathbf{Y} \times \mathbf{X}.$$

For $\mathbf{e} : \mathbf{V} \to \mathbf{Y}$, $\mathbf{f} : \mathbf{W} \to \mathbf{Y}$, $\mathbf{g} : \mathbf{W} \to \mathbf{Z}$, and $\mathbf{h} : \mathbf{X} \to \mathbf{Z}$ we have

$$\mathbf{V} \times_{\mathbf{e},\mathbf{Y},\mathbf{f}\circ\boldsymbol{\pi}_{\mathbf{W}}} (\mathbf{W} \times_{\mathbf{g},\mathbf{Z},\mathbf{h}} \mathbf{X}) \simeq (\mathbf{V} \times_{\mathbf{e},\mathbf{Y},\mathbf{f}} \mathbf{W}) \times_{\mathbf{g}\circ\boldsymbol{\pi}_{\mathbf{W}},\mathbf{Z},\mathbf{h}} \mathbf{X},$$

so associativity holds without signs.

Derived Differential Geometry

Lecture 11 of 14: Derived manifolds and orbifolds with boundary and with corners

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Summer 2015

These slides, and references, etc., available at
http://people.maths.ox.ac.uk/~joyce/DDG2015

Plan of talk:

11. Derived manifolds with boundary and corners

Manifolds are spaces locally modelled on $\mathbb{R}^n$. Similarly, *manifolds with boundary* are spaces locally modelled on $[0,\infty)\times\mathbb{R}^{n-1}$, and *manifolds with corners* spaces locally modelled on $[0,\infty)^k\times\mathbb{R}^{n-k}$.
We will explain how to define derived manifolds and orbifolds with boundary, and with corners. They form 2-categories $\mathbf{dMan^b},\mathbf{dMan^c}$, $\mathbf{dOrb^b},\mathbf{dOrb^c},\mathbf{Kur^b},\mathbf{Kur^c}$, with $\mathbf{dMan}\subset\mathbf{dMan^b}\subset\mathbf{dMan^c}$, etc.
Derived orbifolds (Kuranishi spaces) with corners are important in symplectic geometry, since moduli spaces of J-holomorphic curves with boundary in a Lagrangian are Kuranishi spaces with corners, and Lagrangian Floer cohomology and Fukaya categories depend on understanding such moduli spaces and their boundaries.
'Things with corners' – even basic questions, like what is a smooth map of manifolds with corners – are often more complicated than you might expect.

11.1. Manifolds with corners

Defining manifolds with corners just as objects is a straightforward generalization of the definition of manifolds.

Definition

Write $\mathbb{R}^n_k=[0,\infty)^k\times\mathbb{R}^{n-k}$ for $0\leqslant k\leqslant n$. Elements of $\mathbb{R}^n_k$ are $(x_1,\ldots,x_n)$ with $x_1,\ldots,x_k\geqslant 0$ and $x_{k+1},\ldots,x_n\in\mathbb{R}$.
A *manifold with corners* of dimension n is a Hausdorff, second countable topological space X equipped with an atlas of charts $\{(V_i,\psi_i):i\in I\}$, where $V_i\subseteq\mathbb{R}^n_k$ is open, and $\psi_i:V_i\rightarrow X$ is a homeomorphism with an open subset $\mathrm{Im}\,\psi_i$ of X for all $i\in I$, and $\psi_j^{-1}\circ\psi_i:\psi_i^{-1}(\mathrm{Im}\,\psi_j)\rightarrow\psi_j^{-1}(\mathrm{Im}\,\psi_i)$ is a diffeomorphism of open subsets of $\mathbb{R}^n_k$ for all $i,j\in I$. (That is, $\psi_j^{-1}\circ\psi_i$ must extend to a diffeomorphism of open subsets of $\mathbb{R}^n$.)

Let $U \subseteq \mathbb{R}^m_k$, $V \subseteq \mathbb{R}^n_l$ be open. There are broadly four sensible definitions of when a continuous map $g : U \to V$ is 'smooth', where $g = (g_1, \ldots, g_n)$ with $g_j = g_j(x_1, \ldots, x_m)$:

(a) We call g *weakly smooth* if there exist an open neighbourhood U' of U in $\mathbb{R}^m$, and a smooth map $g' : U' \to \mathbb{R}^n$ in the usual sense, such that $g'|_U = g$. By Seeley's Extension Theorem, this holds iff all derivatives $\frac{\partial^k g}{\partial x_{i_1} \cdots \partial x_{i_k}}$ exist and are continuous, using one-sided derivatives at boundaries.

(b) (Richard Melrose) We call g *smooth* if it is weakly smooth and locally in U, for each $j = 1, \ldots, l$ we have either (i) $g_j = 0$, or (ii)

$$g_j(x_1, \ldots, x_n) = x_1^{a_{1,j}} \cdots x_k^{a_{k,j}} h_j(x_1, \ldots, x_n), \tag{11.1}$$

for $a_1, \ldots, a_k \in \mathbb{N}$ and $h_j : U \to (0, \infty)$ weakly smooth.

(c) (Melrose) We call smooth g *interior* if (i) does not occur.

(d) (Joyce 2009) We call g *strongly smooth* if it is smooth, and in (11.1) we always have $a_{1,j} + \cdots + a_{k,j} = 0$ or 1.

Derived manifolds with boundary and corners
Bordism, virtual classes, and virtual chains
Manifolds with corners
Derived manifolds and orbifolds with corners

Definition

Let X, Y be manifolds with corners, and $f : X \to Y$ be continuous. We say that f is *weakly smooth*, or *smooth*, or *interior*, or *strongly smooth*, if for all charts (U_i, ϕ_i) on X with $U_i \subseteq \mathbb{R}^m_k$ and (V_j, ϕ_j) on Y with $V_j \subseteq \mathbb{R}^n_l$, the map

$$\psi_j^{-1} \circ f \circ \phi_i : (f \circ \phi_i)^{-1}(\mathrm{Im}\, \psi_j) \longrightarrow V_j$$

is weakly smooth, or smooth, or interior, or strongly smooth, respectively, as a map between open subsets of $\mathbb{R}^m_k$ and $\mathbb{R}^n_l$.

All four classes of maps are closed under composition and include identities, and so make manifolds with corners into a category. My favourite is smooth maps (which Richard Melrose calls 'b-maps'). Write $\mathbf{Man^c}$ for the category with objects manifolds with corners, and morphisms smooth maps.

Example 11.1

(i) $e : \mathbb{R} \to [0, \infty)$, $e(x) = x^2$, is weakly smooth but not smooth.
(ii) $f : [0, \infty) \to [0, \infty)$, $f(x) = x^2$, is smooth and interior, but not strongly smooth.
(iii) $g : [0, \infty)^2 \to [0, \infty)$, $g(x, y) = x + y$, is weakly smooth but not smooth.
(iv) $h : [0, \infty)^2 \to [0, \infty)$, $h(x, y) = xy$, is smooth and interior, but not strongly smooth.
(v) $i : \mathbb{R} \to [0, \infty)$, $i(x) = 0$, is (strongly) smooth but not interior.
(vi) $j : [0, \infty) \to \mathbb{R}$, $j(x) = x$, is (strongly) smooth and interior.

An n-manifold with corners X has a natural *depth stratification* $X = \coprod_{l=0}^{n} S^l(X)$, where $S^l(X)$ is the set of $x \in X$ in a boundary stratum of codimension l. If $(x_1, \ldots, x_n) \in \mathbb{R}^n_k$ are local coordinates on X, then $(x_1, \ldots, x_n)$ lies in $S^l(X)$ (i.e. has *depth* l) iff l out of $x_1, \ldots, x_k$ are zero. Then $S^l(X)$ is a manifold without boundary of dimension $n - l$. We call $S^0(X)$ the *interior* X°. The closure is $\overline{S^l(X)} = \coprod_{k=l}^{n} S^k(X)$. We call X a *manifold with boundary* if $S^l(X) = \emptyset$ for $l > 1$, and a *manifold without boundary* (i.e. an ordinary manifold) if $S^l(X) = \emptyset$ for $l > 0$.
A *local boundary component* β at $x \in X$ is a choice of local connected component of $S^1(X)$ near x. That is, β is a local boundary hypersurface containing x. If $(x_1, \ldots, x_n) \in \mathbb{R}^n_k$ are local coordinates on X, then local boundary components β at $(x_1, \ldots, x_n)$ correspond to a choice of $l = 1, \ldots, k$ such that $x_l = 0$. If $x \in S^k(X)$ then there are k distinct local boundary components $\beta_1, \ldots, \beta_k$ at x.

Boundaries and corners of manifolds with corners

Definition

Let X be a manifold with corners. The *boundary* ∂X, as a set, is $\partial X = \{(x,\beta) : x \in X,\ \beta$ is a local boundary component of X at $x\}$. Define $i_X : \partial X \to X$ by $i_X : (x,\beta) \mapsto x$. Then ∂X has the natural structure of a manifold with corners of dimension $\dim X - 1$, such that i_X is a smooth (but not interior) map. Note that $i_X^{-1}(x)$ is k points if $x \in S^k(X)$. So i_X may not be injective. We have

$$\partial^k X \cong \{(x,\beta_1,\ldots,\beta_k) : x \in X,\ \beta_1,\ldots,\beta_k \text{ are distinct local boundary components of } X \text{ at } x\}.$$

The symmetric group S_k acts on $\partial^k X$ by permuting $\beta_1,\ldots,\beta_k$. The *k-corners* $C_k(X)$ is $\partial^k X/S_k$, a manifold with corners of dimension $\dim X - k$, with $C_0(X) = X$, $C_1(X) = \partial X$. We have

$$C_k(X) \cong \{(x,\{\beta_1,\ldots,\beta_k\}) : x \in X,\ \beta_1,\ldots,\beta_k \text{ are distinct local boundary components of } X \text{ at } x\}.$$

Example 11.2

Consider $X = [0,\infty)^2$ with coordinates (x,y). The local boundary components of X at (x',y') are $\{x=0\}$ if $x'=0$ and $\{y=0\}$ if $y'=0$, so there are two local boundary components $\{x=0\}$, $\{y=0\}$ at $(0,0)$.

The boundary ∂X is $[0,\infty) \amalg [0,\infty)$, two copies of $[0,\infty)$, with the first from local boundary component $\{y=0\}$ with $i_X : x \mapsto (x,0)$, and the second from local boundary component $\{x=0\}$ with $i_X : y \mapsto (0,y)$.

Also $\partial^2 X$ is two points, both mapped to $(0,0)$ by $i_X \circ i_{\partial X}$, and $S_2 = \mathbb{Z}_2$ acts freely on $\partial^2 X$ by swapping the points. $C_2(X) = \partial^2 X/S_2$ is one point.

Note that ∂X and $\partial^2 X$ are not subsets of X, the maps $\partial X \to X$, $\partial^2 X \to X$ are 2:1 over $(0,0)$.

How do smooth maps act on boundaries and corners?

Let $f : X \to Y$ be a smooth map of manifolds with corners. In general there is no natural smooth map $\partial f : \partial X \to \partial Y$ (e.g. for $j : [0,\infty) \to \mathbb{R}$, $j(x) = x$ we have $\partial X = \{0\}$, $\partial Y = \emptyset$, so no map $\partial X \to \partial Y$). So boundaries are not a functor $\partial : \mathbf{Man^c} \to \mathbf{Man^c}$.
Nonetheless boundaries do play nicely with smooth maps (though *not* with weakly smooth maps).
One way to show this is to write $\check{\mathbf{Man}}^{\mathbf{c}}$ for the category of *manifolds with corners of mixed dimension*, with objects $\coprod_{n\geqslant 0} X_n$ for X_n a manifold with corners of dimension n, and morphisms continuous maps $f : \coprod_{m\geqslant 0} X_m \to \coprod_{n\geqslant 0} Y_n$ smooth on each component. Then there is a natural *corner functor* $C : \mathbf{Man^c} \to \check{\mathbf{Man}}^{\mathbf{c}}$, with $C(X) = \coprod_{n=0}^{\dim X} C_k(X)$ on objects, and

$$C(f) : (x, \{\beta_1, \ldots, \beta_k\}) \longmapsto (y, \{\gamma_1, \ldots, \gamma_l\}),$$

where $f(x) = y$, and $\gamma_1, \ldots, \gamma_l$ are all the local boundary components of Y at y which contain $f(\beta_i)$ for all $i = 1, \ldots, k$.

We call a smooth map $f : X \to Y$ *simple* if $C(f)$ maps $C_k(X) \to C_k(Y)$ for all $k \geqslant 0$. This is a discrete condition on f. Diffeomorphisms are simple. Simple maps are closed under composition and include identities, so they give a subcategory $\mathbf{Man^c_{si}} \subset \mathbf{Man^c}$. Then boundaries and corners give functors

$$\partial : \mathbf{Man^c_{si}} \longrightarrow \mathbf{Man^c_{si}}, \quad C_k : \mathbf{Man^c_{si}} \longrightarrow \mathbf{Man^c_{si}},$$

mapping $X \mapsto \partial X$, $X \mapsto C_k(X)$ on objects, and $f \mapsto C(f)|_{C_1(X)}$, $f \mapsto C(f)|_{C_k(X)}$ on morphisms $f : X \to Y$.

Tangent bundles and b-tangent bundles

For manifolds with corners X there are *two different* notions of tangent bundle: the *tangent bundle* TX, and the *b-tangent bundle* ${}^bTX$ (due to Richard Melrose). If $(x_1,\ldots,x_n)\in\mathbb{R}^n_k$ are local coordinates on X, with $x_1,\ldots,x_k\geqslant 0$, then

$$TX=\left\langle\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}\right\rangle,\quad {}^bTX=\left\langle x_1\frac{\partial}{\partial x_1},\ldots,x_k\frac{\partial}{\partial x_k},\frac{\partial}{\partial x_{k+1}},\ldots,\frac{\partial}{\partial x_n}\right\rangle.$$

Think of $C^\infty(TX)$ as the vector fields v on X, and $C^\infty({}^bTX)$ as the vector fields v on X which are tangent to each boundary stratum. Usually it is better to use ${}^bTX$ than TX.

You can only define the b-derivative ${}^bTf:{}^bTX\to f^*({}^bTY)$ if $f:X\to Y$ is an *interior* map of manifolds with corners.

There are also two cotangent bundles $T^*X=(TX)^*$ and ${}^bT^*X=({}^bTX)^*$, where in coordinates

$${}^bT^*X=\left\langle x_1^{-1}\mathrm{d}x_1,\ldots,x_k^{-1}\mathrm{d}x_k,\mathrm{d}x_{k+1},\ldots,\mathrm{d}x_n\right\rangle.$$

Transverse fibre products?

Good conditions for existence of fibre products of $g:X\to Z$, $h:Y\to Z$ in $\mathbf{Man^c}$ are complicated. Copying the usual definition of transversality using either TX,TY,TZ or ${}^bTX,{}^bTY,{}^bTZ$ is not enough: you need additional discrete conditions on how g,h act on $\partial^jX,\partial^kY,\partial^lZ$. See Joyce arXiv:0910.3518 for sufficient conditions for existence of fibre products in $\mathbf{Man^c_{ss}}$, the category of manifolds with corners and strongly smooth maps.

The nicest answer (Joyce arXiv:1501.00401) is to define *manifolds with generalized corners* (*g-corners*) $\mathbf{Man^{gc}}$, allowing more complicated local models for corners than $[0,\infty)^k\times\mathbb{R}^{n-k}$. Then b-tangent bundles ${}^bTX$ and interior maps $g:X\to Z$ make sense for X in $\mathbf{Man^{gc}}$. If $g:X\to Z$ and $h:Y\to Z$ are *b-transverse* interior maps in $\mathbf{Man^{gc}}$ (i.e. ${}^bT_xg\oplus{}^bT_yh:{}^bT_xX\oplus{}^bT_yY\to{}^bT_zZ$ is always surjective) then a fibre product $X\times_{g,Z,h}Y$ exists in $\mathbf{Man^{gci}}$, the category of manifolds with g-corners and interior

11.2 Derived manifolds and orbifolds with corners

How should we define derived manifolds and orbifolds with boundary and corners?

My d-manifolds book gives definitions of strict 2-categories $\mathbf{dMan^b}, \mathbf{dMan^c}$, $\mathbf{dOrb^b}, \mathbf{dOrb^c}$ of d-manifolds and d-orbifolds with boundary and corners using C^∞-algebraic geometry. They are full 2-subcategories of strict 2-categories $\mathbf{dSpa^b}, \mathbf{dSpa^c}$, $\mathbf{dSta^b}, \mathbf{dSta^c}$ of d-spaces and d-stacks with boundary and corners, which are classes of derived C^∞-schemes and Deligne–Mumford C^∞-stacks with boundary and corners. The details are complex and messy.

However, a far less painful route is to start with my definition of (M-)Kuranishi spaces (arXiv:1409.6908), and take the V in (M-)Kuranishi neighbourhoods (V, E, s, ψ) or (V, E, Γ, s, ψ) to be manifolds with boundary, or with corners. We replace manifolds by manifolds with boundary or (g-)corners throughout, and make a few other changes (use interior/simple maps, use TV or ${}^bTV$, etc.)

Thus, we can define ordinary categories $\mathbf{MKur^b}, \mathbf{MKur^c}$ of *M-Kuranishi spaces with boundary and corners*, with $\mathbf{MKur} \subset \mathbf{MKur^b} \subset \mathbf{MKur^c}$, a kind of derived manifold with boundary, or corners.

We also have weak 2-categories $\mathbf{Kur^b}, \mathbf{Kur^c}$ of *Kuranishi spaces with boundary and corners*, and $\mathbf{Kur^{gc}}$ of *Kuranishi spaces with generalized corners* (*g-corners*), with $\mathbf{Kur} \subset \mathbf{Kur^b} \subset \mathbf{Kur^c} \subset \mathbf{Kur^{gc}}$. These are forms of derived orbifolds with boundary, or corners, or g-corners.

Write $\mathbf{Kur^b_{trG}} \subset \mathbf{Kur^b}$, $\mathbf{Kur^c_{trG}} \subset \mathbf{Kur^c}$, $\mathbf{Kur^{gc}_{trG}} \subset \mathbf{Kur^{gc}}$ for the full 2-subcategories of $\mathbf{X}$ with orbifold groups $G_x\mathbf{X} = \{1\}$ for all $x \in \mathbf{X}$. Then $\mathbf{Kur^b_{trG}}, \mathbf{Kur^c_{trG}}, \mathbf{Kur^{gc}_{trG}}$ are 2-categories of derived manifolds with boundary, or corners, or g-corners.

Orbifold groups, (b-)tangent and (b-)obstruction spaces

As for Kuranishi spaces in §9.1, Kuranishi spaces $\mathbf{X}$ with boundary or corners have *orbifold groups* $G_x\mathbf{X}$, *tangent spaces* $T_x\mathbf{X}$, and *obstruction spaces* $O_x\mathbf{X}$ with $\dim T_x\mathbf{X} - \dim O_x\mathbf{X} = \mathrm{vdim}\,\mathbf{X}$, and for 1-morphisms $\mathbf{f}:\mathbf{X}\to\mathbf{Y}$ and $x\in\mathbf{X}$ with $\mathbf{f}(x)=y\in\mathbf{Y}$ we have morphisms $G_x\mathbf{f}:G_x\mathbf{X}\to G_y\mathbf{Y}$, $T_x\mathbf{f}:T_x\mathbf{X}\to T_y\mathbf{Y}$, $O_x\mathbf{f}:O_x\mathbf{X}\to O_y\mathbf{Y}$, and for 2-morphisms $\boldsymbol{\eta}:\mathbf{f}\Rightarrow\mathbf{g}$ we have $G_x\boldsymbol{\eta}\in G_y\mathbf{Y}$.

Just as manifolds with corners X have two tangent bundles $TX, {}^bTX$, and it is usually better to use ${}^bTX$, so Kuranishi spaces $\mathbf{X}$ with boundary or corners also have *b-tangent spaces* ${}^bT_x\mathbf{X}$, and *b-obstruction spaces* ${}^bO_x\mathbf{X}$ with $\dim {}^bT_x\mathbf{X} - \dim {}^bO_x\mathbf{X} = \mathrm{vdim}\,\mathbf{X}$.

If $\mathbf{f}:\mathbf{X}\to\mathbf{Y}$ is an *interior* 1-morphism and $x\in\mathbf{X}$ with $\mathbf{f}(x)=y\in\mathbf{Y}$ we have ${}^bT_x\mathbf{f}:{}^bT_x\mathbf{X}\to{}^bT_y\mathbf{Y}$, ${}^bO_x\mathbf{f}:{}^bO_x\mathbf{X}\to{}^bO_y\mathbf{Y}$.

It is usually better to use ${}^bT_x\mathbf{X}, {}^bO_x\mathbf{X}$ rather than $T_x\mathbf{X}, O_x\mathbf{X}$.

For Kuranishi spaces $\mathbf{X}$ with g-corners we define ${}^bT_x\mathbf{X}, {}^bO_x\mathbf{X}$ but not $T_x\mathbf{X}, O_x\mathbf{X}$.

If $(V_i,E_i,\Gamma_i,s_i,\psi_i)$ is a Kuranishi neighbourhood in the Kuranishi structure on $\mathbf{X}$, so that V_i is a manifold with corners, and $x\in\mathrm{Im}\,\psi_i\subseteq\mathbf{X}$ with $x=\psi_i(v\Gamma_i)$ for $v\in s_i^{-1}(0)\subseteq V_i$, then $T_x\mathbf{X}, O_x\mathbf{X}, {}^bT_x\mathbf{X}, {}^bO_x\mathbf{X}$ are defined by the exact sequences

$$\begin{tikzcd}
0 \arrow[r] & {}^bT_x\mathbf{X} \arrow[r] \arrow[d, dotted] & {}^bT_vV_i \arrow[r, "{}^b\mathrm{d}s_i|_v"'] \arrow[d, dotted] & E_i|_v \arrow[r] \arrow[d, dotted, "\mathrm{id}"] & {}^bO_x\mathbf{X} \arrow[r] \arrow[d, dotted] & 0 \\
0 \arrow[r] & T_x\mathbf{X} \arrow[r] & T_vV_i \arrow[r, "\mathrm{d}s_i|_v"] & E_i|_v \arrow[r] & O_x\mathbf{X} \arrow[r] & 0,
\end{tikzcd}$$

where there are natural vertical morphisms making the diagram commute.

Boundaries of Kuranishi spaces with corners

If $\mathbf{X}$ is a Kuranishi space with corners, we can define a natural Kuranishi space with corners $\partial\mathbf{X}$ called the *boundary* of $\mathbf{X}$, with a morphism $\mathbf{i}_{\mathbf{X}}:\partial\mathbf{X}\to\mathbf{X}$, and with $\mathrm{vdim}\,\partial\mathbf{X}=\mathrm{vdim}\,\mathbf{X}-1$. If $(V_i,E_i,\Gamma_i,s_i,\psi_i)_{i\in I}$ are the Kuranishi neighbourhoods in the Kuranishi structure on $\mathbf{X}$, then $(\partial V_i,E_i|_{\partial V_i},\Gamma_i,s_i|_{\partial V_i},\psi_i')_{i\in I}$ are the Kuranishi neighbourhoods in the Kuranishi structure on $\partial\mathbf{X}$.

If $\mathbf{X}$ is a Kuranishi space with boundary, then $\partial\mathbf{X}$ is a Kuranishi space without boundary. If $\mathbf{X}$ is a Kuranishi space without boundary, then $\partial\mathbf{X}=\emptyset$.

There is a strict action of the symmetric group S_k on $\partial^k\mathbf{X}$, which is free if $\mathbf{X}$ has trivial orbifold groups. Then the *k-corners* $C_k(\mathbf{X})=(\partial^k\mathbf{X})/S_k$ is a Kuranishi space with corners for $k\geqslant 0$, with $\mathrm{vdim}\,C_k(\mathbf{X})=\mathrm{vdim}\,\mathbf{X}-k$.

Write $\check{\mathbf{Kur}}^{\mathbf{c}}$ for the 2-category of *Kuranishi spaces of mixed dimension*, with objects $\coprod_{n\in\mathbb{Z}}\mathbf{X}_n$ for $\mathbf{X}_n$ a Kuranishi space with corners of virtual dimension $n\in\mathbb{Z}$, and 1- and 2-morphisms $\mathbf{f},\mathbf{g}:\coprod_{m\in\mathbb{Z}}\mathbf{X}_m\to\coprod_{n\in\mathbb{Z}}\mathbf{Y}_n$, $\boldsymbol{\eta}:\mathbf{f}\Rightarrow\mathbf{g}$ being 1- and 2-morphisms in $\mathbf{Kur^c}$ on each component.

Then there is a natural *corner 2-functor* $C:\mathbf{Kur^c}\to\check{\mathbf{Kur}}^{\mathbf{c}}$, with $C(\mathbf{X})=\coprod_{n\geqslant 0}C_k(\mathbf{X})$ on objects.

Call a 1-morphism $\mathbf{f}:\mathbf{X}\to\mathbf{Y}$ in $\mathbf{Kur^c}$ *simple* if $C(\mathbf{f})$ maps $C_k(\mathbf{X})\to C_k(\mathbf{Y})$ for all $k\geqslant 0$. Simple 1-morphisms are closed under composition and include identities, so they define a 2-subcategory $\mathbf{Kur^c_{si}}\subset\mathbf{Kur^c}$. Then we have 2-functors $\partial, C_k:\mathbf{Kur^c_{si}}\to\mathbf{Kur^c_{si}}$ mapping $\mathbf{X}\mapsto\partial\mathbf{X}$, $\mathbf{X}\mapsto C_k(\mathbf{X})$ on objects, and $\mathbf{f}\mapsto C(\mathbf{f})|_{C_1(\mathbf{X})}$, $\mathbf{f}\mapsto C(\mathbf{f})|_{C_k(\mathbf{X})}$ on 1-morphisms $\mathbf{f}:\mathbf{X}\to\mathbf{Y}$, and $\boldsymbol{\eta}\mapsto C(\boldsymbol{\eta})|_{C_1(\mathbf{X})}$, $\boldsymbol{\eta}\mapsto C(\boldsymbol{\eta})|_{C_k(\mathbf{X})}$ on 2-morphisms $\boldsymbol{\eta}:\mathbf{f}\Rightarrow\mathbf{g}$.

Any equivalence $\mathbf{f}:\mathbf{X}\to\mathbf{Y}$ in $\mathbf{Kur^c}$ is simple, and then $\partial\mathbf{f}:\partial\mathbf{X}\to\partial\mathbf{Y}$ and $C_k(\mathbf{f}):C_k(\mathbf{X})\to C_k(\mathbf{Y})$ are equivalences. So boundaries $\partial\mathbf{X}$ and corners $C_k(\mathbf{X})$ are preserved by equivalences.

Differential geometry of Kuranishi spaces with corners

All the material in lectures 9 and 10 on differential geometry of derived manifolds and orbifolds extends to the corners case, with suitable modifications: immersions, embeddings and derived submanifolds; embedding derived manifolds into manifolds; submersions; orientations; and transverse fibre products.
We give some highlights. Here is an analogue of Corollary 9.8:

Theorem 11.3

A d-manifold with corners $\mathbf{X}$ is equivalent in $\mathbf{dMan^c}$ to a standard model d-manifold with corners $\mathbf{S}_{V,E,s}$ if and only if $\dim T_x\mathbf{X} + \lvert \mathbf{i}_{\mathbf{X}}^{-1}(x)\rvert$ is bounded above for all $x \in \mathbf{X}$. This always holds if $\mathbf{X}$ is compact.

Here $\lvert \mathbf{i}_{\mathbf{X}}^{-1}(x)\rvert$ is the 'depth' of $x \in \mathbf{X}$, the codimension of the boundary stratum it lives in.

W-submersions and submersions

We can define *w-submersions* and *submersions* $f : \mathbf{X} \to \mathbf{Y}$ of derived manifolds and orbifolds with corners. The definition includes a discrete condition on how $\mathbf{f}$ acts on $\partial^k\mathbf{X}, \partial^l\mathbf{Y}$.
Here are analogues of Theorems 9.11 and 9.12.

Theorem 11.4

Suppose $\mathbf{g} : \mathbf{X} \to \mathbf{Z}$ is a w-submersion in $\mathbf{dMan^c}$. Then for any 1-morphism $\mathbf{h} : \mathbf{Y} \to \mathbf{Z}$ in $\mathbf{dMan^c}$, the fibre product $\mathbf{X} \times_{\mathbf{g},\mathbf{Z},\mathbf{h}} \mathbf{Y}$ exists in $\mathbf{dMan^c}$.

Theorem 11.5

Suppose $\mathbf{g} : \mathbf{X} \to \mathbf{Z}$ is a submersion in $\mathbf{dMan^c}$. Then for any 1-morphism $\mathbf{h} : Y \to \mathbf{Z}$ in $\mathbf{dMan}$ with Y a manifold with corners, the fibre product $\mathbf{X} \times_{\mathbf{g},\mathbf{Z},\mathbf{h}} Y$ exists in $\mathbf{dMan^c}$ and is a manifold with corners. In particular, the fibres $\mathbf{X}_z = \mathbf{X} \times_{\mathbf{g},\mathbf{Z},z} *$ of $\mathbf{g}$ for $z \in \mathbf{Z}$ are manifolds with corners.

Orientations

Here is the analogue of Theorem 9.14:

Theorem 11.6

(a) *Every d-manifold with corners* $\mathbf{X}$ *has a* **canonical bundle** $K_{\mathbf{X}}$*, a* C^∞ *real line bundle over the* C^∞*-scheme* $\underline{X}$*, natural up to canonical isomorphism, with* $K_{\mathbf{X}}|_x \cong \Lambda^{\rm top}({}^bT^*_x\mathbf{X}) \otimes \Lambda^{\rm top}({}^bO_x\mathbf{X})$ *for* $x \in \mathbf{X}$.

(b) *If* $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ *is an étale 1-morphism (e.g. an equivalence), there is a canonical, functorial isomorphism* $K_{\mathbf{f}}: K_{\mathbf{X}} \to \underline{f}^*(K_{\mathbf{Y}})$. *If* $\mathbf{f}, \mathbf{g}: \mathbf{X} \to \mathbf{Y}$ *are 2-isomorphic then* $K_{\mathbf{f}} = K_{\mathbf{g}}$.

(c) *If* $\mathbf{X} \simeq \mathbf{S}_{V,E,s}$*, there is a canonical isomorphism*

$$K_{\mathbf{X}} \cong \left(\Lambda^{\dim V}({}^bT^*V) \otimes \Lambda^{\mathrm{rank}\,E} E\right)|_{s^{-1}(0)}.$$

(d) *There is a canonical isomorphism* $K_{\partial\mathbf{X}} \cong \underline{i}_{\mathbf{X}}^*(K_{\mathbf{X}})$.

Analogues of **(a)**–**(d)** *hold for d-orbifolds and Kuranishi spaces with corners.*

We then define an *orientation* of a derived manifold or orbifold with corners $\mathbf{X}$ to be an orientation on the fibres of $K_{\mathbf{X}}$. Note that Theorem 11.6(d) implies that if $\mathbf{X}$ is oriented then so are $\partial\mathbf{X}, \partial^2\mathbf{X}, \partial^3\mathbf{X}, \ldots$.
Recall that the k-corners $C_k(\mathbf{X})$ is $C_k(\mathbf{X}) = \partial^k\mathbf{X}/S_k$. If $k \geqslant 2$ then the action of S_k on $\partial^k\mathbf{X}$ is not orientation preserving, and $C_k(\mathbf{X})$ may not be orientable. This also happens for ordinary manifolds with corners.

Transverse fibre products

In the 2-categories $\mathbf{dMan^c}, \mathbf{dOrb^c}, \mathbf{Kur^c}$, fibre products of $\mathbf{g}:\mathbf{X}\to\mathbf{Z}$, $\mathbf{h}:\mathbf{Y}\to\mathbf{Z}$ exist provided a 'd-transversality' condition holds – $O_x\mathbf{g}\oplus O_y\mathbf{h}:O_x\mathbf{X}\oplus O_y\mathbf{Y}\to O_z\mathbf{Z}$ or ${}^bO_x\mathbf{g}\oplus {}^bO_y\mathbf{h}:{}^bO_x\mathbf{X}\oplus {}^bO_y\mathbf{Y}\to {}^bO_z\mathbf{Z}$ is surjective – *and* a discrete condition ('b-transversality' or 'c-transversality') on the action of $\mathbf{g},\mathbf{h}$ on corners holds.

If $\mathbf{Z}$ is a manifold without boundary both of these are trivial, giving an analogue of Corollary 10.3:

Theorem 11.7

Suppose $\mathbf{g}:\mathbf{X}\to\mathbf{Z}$, $\mathbf{h}:\mathbf{Y}\to\mathbf{Z}$ *are* 1*-morphisms in* $\mathbf{dMan^c}$, *with* $\mathbf{Z}$ *a manifold without boundary. Then a fibre product* $\mathbf{W}=\mathbf{X}\times_{\mathbf{g},\mathbf{Z},\mathbf{h}}\mathbf{Y}$ *exists in* $\mathbf{dMan^c}$, *with* $\mathrm{vdim}\,\mathbf{W}=\mathrm{vdim}\,\mathbf{X}+\mathrm{vdim}\,\mathbf{Y}-\dim\mathbf{Z}$.

We can dispense with these discrete conditions on corners if we work with 'generalized corners'. Then we have:

Theorem 11.8

Let $\mathbf{g}:\mathbf{X}\to\mathbf{Z}$, $\mathbf{h}:\mathbf{Y}\to\mathbf{Z}$ *be interior* 1*-morphisms in* $\mathbf{Kur^{gc}}$. *Suppose* $\mathbf{g},\mathbf{h}$ *are* **d-transverse**, *that is,* ${}^bO_x\mathbf{g}\oplus(\gamma\cdot{}^bO_y\mathbf{h}):{}^bO_x\mathbf{X}\oplus{}^bO_y\mathbf{Y}\to{}^bO_z\mathbf{Z}$ *is surjective for all* $x\in\mathbf{X}$, $y\in\mathbf{Y}$ *with* $\mathbf{g}(x)=\mathbf{h}(y)=z\in\mathbf{Z}$ *and* $\gamma\in G_z\mathbf{Z}$. *Then a fibre product* $\mathbf{W}=\mathbf{X}\times_{\mathbf{g},\mathbf{Z},\mathbf{h}}\mathbf{Y}$ *exists in the 2-category* $\mathbf{Kur^{gc}_{in}}$ *of Kuranishi spaces with g-corners and interior* 1*-morphisms, with* $\mathrm{vdim}\,\mathbf{W}=\mathrm{vdim}\,\mathbf{X}+\mathrm{vdim}\,\mathbf{Y}-\dim\mathbf{Z}$.

For any such d-transverse fibre product $\mathbf{W}=\mathbf{X}\times_{\mathbf{g},\mathbf{Z},\mathbf{h}}\mathbf{Y}$ in $\mathbf{dMan^c}, \mathbf{dOrb^c}, \mathbf{Kur^c}, \mathbf{Kur^{gc}_{in}}$, if $\mathbf{X},\mathbf{Y},\mathbf{Z}$ are oriented, then $\mathbf{W}$ is oriented.

Derived Differential Geometry

Lecture 12 of 14: Bordism, virtual classes, and virtual chains

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Summer 2015

These slides, and references, etc., available at
http://people.maths.ox.ac.uk/∼joyce/DDG2015

Plan of talk:

12. Bordism, virtual classes, and virtual chains

In many important areas of geometry to do with enumerative invariants (e.g. Donaldson and Seiberg–Witten invariants of 4-manifolds, Gromov–Witten invariants of symplectic manifolds, Donaldson–Thomas invariants of Calabi–Yau 3-folds, ...), we form a moduli space $\bar{\mathcal{M}}$ with some geometric structure, and we want to 'count' $\bar{\mathcal{M}}$ to get a number in $\mathbb{Z}$ or $\mathbb{Q}$ (if $\bar{\mathcal{M}}$ has no boundary and dimension 0), or a homology class ('virtual class') $[\bar{\mathcal{M}}]_{\rm virt}$ in some homology theory (if $\bar{\mathcal{M}}$ has no boundary and dimension > 0).
For more complicated theories (Floer homology, Fukaya categories), $\bar{\mathcal{M}}$ has boundary, and then we must define a chain $[\bar{\mathcal{M}}]_{\rm virt}$ in the chain complex (C_*, ∂) of some homology theory (a 'virtual chain'), where ideally we want $\partial[\bar{\mathcal{M}}]_{\rm virt} = [\partial\bar{\mathcal{M}}]_{\rm virt}$.

In general $\bar{\mathcal{M}}$ is not a manifold (or orbifold). However, the point is to treat $\bar{\mathcal{M}}$ *as if* it were a compact, oriented manifold, so that in particular, if $\partial\bar{\mathcal{M}} = \emptyset$ then $\bar{\mathcal{M}}$ has a fundamental class $[\bar{\mathcal{M}}]$ in the homology group $H_{\dim\bar{\mathcal{M}}}(\bar{\mathcal{M}};\mathbb{Z})$.
All of these 'counting invariant' theories over $\mathbb{R}$ or $\mathbb{C}$, in both differential and algebraic geometry, can be understood using derived differential geometry. The point is that the moduli spaces $\bar{\mathcal{M}}$ should be compact, oriented derived manifolds or orbifolds (possibly with corners). Then we show that compact, oriented derived manifolds or orbifolds (with corners) have virtual classes (virtual chains), and these are used to define the invariants.
There is an easy way to define virtual classes for compact, oriented derived manifolds without boundary, using *bordism*, so we explain this first. It does not work very well in the orbifold case, though.

12.1 Bordism and derived bordism

Let Y be a manifold. Define the *bordism group* $B_k(Y)$ to have elements $\sim$-equivalence classes $[X, f]$ of pairs (X, f), where X is a compact oriented k-manifold without boundary and $f : X \to Y$ is smooth, and $(X, f) \sim (X', f')$ if there exists a $(k+1)$-manifold with boundary W and a smooth map $e : W \to Y$ with $\partial W \cong X \amalg -X'$ and $e|_{\partial W} \cong f \amalg f'$. It is an abelian group, with addition $[X, f] + [X', f'] = [X \amalg X', f \amalg f']$.

If Y is oriented of dimension n, there is a supercommutative, associative *intersection product* $\bullet : B_k(Y) \times B_l(Y) \to B_{k+l-n}(Y)$ given by $[X, f] \bullet [X', f'] = [X \times_{f,Y,f'} X', \pi_Y]$, choosing X, f, X', f' in their bordism classes with $f : X \to Y$, $f' : X' \to Y$ transverse.

Bordism is a *generalized homology theory*, i.e. it satisfies all the Eilenberg–Steenrod axioms except the Dimension Axiom.

There is a natural morphism $\Pi_{\rm bo}^{\rm hom} : B_k(Y) \to H_k(Y, \mathbb{Z})$ given by $\Pi_{\rm bo}^{\rm hom} : [X, f] \mapsto f_*([X])$, for $[X] \in H_k(X, \mathbb{Z})$ the fundamental class.

Similarly, define the *derived bordism group* $dB_k(Y)$ to have elements $\approx$-equivalence classes $[\mathbf{X}, \mathbf{f}]$ of pairs $(\mathbf{X}, \mathbf{f})$, where $\mathbf{X}$ is a compact oriented d-manifold with $\mathrm{vdim}\,\mathbf{X} = k$ and $\mathbf{f} : \mathbf{X} \to Y$ is a 1-morphism in $\mathbf{dMan}$, and $(\mathbf{X}, \mathbf{f}) \approx (\mathbf{X}', \mathbf{f}')$ if there exists a d-manifold with boundary $\mathbf{W}$ with $\mathrm{vdim}\,\mathbf{W} = k+1$ and a 1-morphism $\mathbf{e} : \mathbf{W} \to \mathbf{Y}$ in $\mathbf{dMan^b}$ with $\partial\mathbf{W} \simeq \mathbf{X} \amalg -\mathbf{X}'$ and $\mathbf{e}|_{\partial\mathbf{W}} \cong \mathbf{f} \amalg \mathbf{f}'$. It is an abelian group, with $[\mathbf{X}, \mathbf{f}] + [\mathbf{X}', \mathbf{f}'] = [\mathbf{X} \amalg \mathbf{X}, \mathbf{f} \amalg \mathbf{f}']$.

If Y is oriented of dimension n, there is a supercommutative, associative *intersection product* $\bullet : dB_k(Y) \times dB_l(Y) \to dB_{k+l-n}(Y)$ given by $[\mathbf{X}, \mathbf{f}] \bullet [\mathbf{X}', \mathbf{f}'] = [\mathbf{X} \times_{\mathbf{f},Y,\mathbf{f}'} \mathbf{X}', \boldsymbol{\pi}_Y]$, with no transversality condition on $\mathbf{X}, \mathbf{f}$, $\mathbf{X}', \mathbf{f}'$.

There is a morphism $\Pi_{\rm bo}^{\rm dbo}: B_k(Y)\to dB_k(Y)$ mapping $[X,f]\mapsto[X,f]$.

Theorem 12.1

$\Pi_{\rm bo}^{\rm dbo}: B_k(Y)\to dB_k(Y)$ is an isomorphism for all k, with $dB_k(Y)=0$ for $k<0$.

This holds as every d-manifold can be perturbed to a manifold.

Composing $(\Pi_{\rm bo}^{\rm dbo})^{-1}$ with $\Pi_{\rm bo}^{\rm hom}: B_k(Y)\to H_k(Y,\mathbb{Z})$ gives a morphism $\Pi_{\rm dbo}^{\rm hom}: dB_k(Y)\to H_k(Y,\mathbb{Z})$. We can interpret this as a *virtual class map* for compact, oriented d-manifolds. In particular, this is an easy proof that *the geometric structure on d-manifolds is strong enough to define virtual classes*.
We can also define *orbifold bordism* $B_k^{\rm orb}(Y)$ and *derived orbifold bordism* $dB_k^{\rm orb}(Y)$, replacing (derived) manifolds by (derived) orbifolds. However, the natural morphism $B_k^{\rm orb}(Y)\to dB_k^{\rm orb}(Y)$ is not an isomorphism, as derived orbifolds cannot always be perturbed to orbifolds (Lemma 9.1).

A virtual class for X in the homology of X?

In algebraic geometry, given a moduli space $\bar{\mathcal{M}}$, it is usual to define the virtual class in the (Chow) homology $H_{{\rm vdim}\,\bar{\mathcal{M}}}(\bar{\mathcal{M}};\mathbb{Q})$. But in differential geometry, given $\bar{\mathcal{M}}$, usually we find a manifold Y with a map $\bar{\mathcal{M}}\to Y$, and define the virtual class $[\bar{\mathcal{M}}]_{\rm virt}$ in the (ordinary) homology $H_{{\rm vdim}\,\bar{\mathcal{M}}}(Y;\mathbb{Q})$. This is because differential-geometric techniques for defining $[\bar{\mathcal{M}}]_{\rm virt}$ involve perturbing $\bar{\mathcal{M}}$, which changes it as a topological space.

Example 12.2

Define $f:\mathbb{R}\to\mathbb{R}$ by $f(x)=e^{-x^{-2}}\sin(\pi/x)$ for $x\neq 0$, and $f(0)=0$. Then f is smooth. Define $\mathbf{X}=\mathbb{R}\times_{f,\mathbb{R},0}*$. Then $\mathbf{X}$ is a compact, oriented derived manifold without boundary, with ${\rm vdim}\,\mathbf{X}=0$. As a topological space we have

$$X=\{1/n:0\neq n\in\mathbb{Z}\}\amalg\{0\}.$$

Then *no virtual class exists* for $\mathbf{X}$ in ordinary homology $H_0(X;\mathbb{Z})$.

Virtual classes in Steenrod or Čech homology

Steenrod homology $H_*^{\rm St}(X;\mathbb{Z})$ (see J. Milnor, *'On the Steenrod homology theory'*, Milnor collected works IV, 2009) is a homology theory of topological spaces. For nice topological spaces X (e.g. manifolds, or finite simplicial complexes) it equals ordinary (e.g. singular) homology $H_*(X;\mathbb{Z})$. It has a useful limiting property:

Theorem 12.3

Let X be a compact subset of a metric space Y, and suppose $W_1, W_2, \ldots$ are open neighbourhoods of X in Y with $\bigcap_{n\geqslant 1} W_n = X$ and $W_1 \supseteq W_2 \supseteq \cdots$. Then $H_k^{\rm St}(X;\mathbb{Z}) \cong \varprojlim_{n\geqslant 1} H_k^{\rm St}(W_n;\mathbb{Z})$.

Čech homology $\check{H}_*(X;\mathbb{Q})$ over $\mathbb{Q}$ has the same property. Singular homology does not.

Following an idea due to Dusa McDuff, we can use this to define a virtual class $[\mathbf{X}]_{\rm virt}$ for a compact oriented d-manifold $\mathbf{X}$ in $H^{\rm St}_{\mathrm{vdim}\,\mathbf{X}}(X;\mathbb{Z})$ or $\check{H}_*(X;\mathbb{Q})$. We may write $\mathbf{X} \simeq \mathbf{S}_{V,E,s}$ by Corollary 9.8. This gives a homeomorphism $X \cong s^{-1}(0)$, for $s^{-1}(0)$ a compact subset of V. Choose open neighbourhoods $W_1, W_2, \ldots$ of $s^{-1}(0)$ in V with $\bigcap_{n\geqslant 1} W_n = s^{-1}(0)$ and $W_1 \supseteq W_2 \supseteq \cdots$. The inclusion $\mathbf{i}_n : \mathbf{X} \hookrightarrow W_n$ defines a d-bordism class $[\mathbf{X}, \mathbf{i}_n] \in dB_{\mathrm{vdim}\,\mathbf{X}}(W_n)$, and hence a homology class $\Pi^{\rm hom}_{\rm dbo}([\mathbf{X}, \mathbf{i}_n])$ in $H_{\mathrm{vdim}\,\mathbf{X}}(W_n;\mathbb{Z}) \cong H^{\rm St}_{\mathrm{vdim}\,\mathbf{X}}(W_n;\mathbb{Z})$. These are preserved by the inclusions $W_{n+1} \hookrightarrow W_n$, and so define a class in the inverse limit $\varprojlim_{n\geqslant 1} H_k^{\rm St}(W_n;\mathbb{Z})$, and thus, by Theorem 12.3, a virtual class $[\mathbf{X}]_{\rm virt}$ in $H^{\rm St}_{\mathrm{vdim}\,\mathbf{X}}(X;\mathbb{Z})$ or $\check{H}_{\mathrm{vdim}\,\mathbf{X}}(X;\mathbb{Q})$.

12.2. Virtual classes for derived orbifolds

If $\mathbf{X}$ is a compact, oriented derived orbifold (d-orbifold or Kuranishi space), Y is a manifold, and $\mathbf{f} : \mathbf{X} \to Y$ is a 1-morphism, then we can define a *virtual class* $[\mathbf{X}]_{\mathrm{virt}}$ in $H_{\mathrm{vdim}\,\mathbf{X}}(Y;\mathbb{Q})$. In the orbifold case it is necessary to work in homology over $\mathbb{Q}$ rather than $\mathbb{Z}$, as points $x \in \mathbf{X}$ with orbifold group $G_x\mathbf{X}$ must be 'counted' with weight $1/|G_x\mathbf{X}|$. There is a standard method for doing this, developed by Fukaya and Ono 1999, in their definition of Gromov–Witten invariants. It is rather messy.
Alternatively, following McDuff–Wehrheim, we can forget Y and define the virtual class $[\mathbf{X}]_{\mathrm{virt}}$ in Čech homology $\check{H}_{\mathrm{vdim}\,\mathbf{X}}(X;\mathbb{Q})$.

The Fukaya–Ono method is to cover $\mathbf{X}$ by finitely many Kuranishi neighbourhoods $(V_i, E_i, \Gamma_i, s_i, \psi_i)$ for $i \in I$ with coordinate changes $\Phi_{ij} : (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (V_j, E_j, \Gamma_j, s_j, \psi_j)$ on overlaps between them in a particularly nice form (a 'good coordinate system'), and compatible maps $f_i : V_i \to Y$ representing $\mathbf{f} : \mathbf{X} \to Y$.
Ideally we would like to choose small perturbations $\tilde{s}_i$ of $s_i : V_i \to E_i$, such that $\tilde{s}_i$ is transverse and Γ_i-equivariant, and Φ_{ij} maps $\tilde{s}_i \to \tilde{s}_j$. Then we could glue the orbifolds $\tilde{s}_i^{-1}(0)/\Gamma_i$ for $i \in I$ using Φ_{ij} to get a compact oriented orbifold $\tilde{\mathfrak{X}}$ with morphism $\tilde{\mathfrak{f}} : \tilde{\mathfrak{X}} \to Y$, and we would set $[\mathbf{X}]_{\mathrm{virt}} = \tilde{\mathfrak{f}}_*([\tilde{\mathfrak{X}}])$, for $[\tilde{\mathfrak{X}}] \in H_{\dim \tilde{\mathfrak{X}}}(\tilde{X};\mathbb{Q})$ the fundamental class of $\tilde{\mathfrak{X}}$.
However, it is generally not possible to find perturbations $\tilde{s}_i$ which are both transverse to the zero section of $E_i \to V_i$, and Γ_i-equivariant.

Instead, we take the perturbations $\tilde{s}_i$ to be 'multisections', $\mathbb{Q}$-weighted mutivalued C^∞-sections of E_i, where the sum of the $\mathbb{Q}$-weights of the branches is 1. As Γ_i can permute the 'branches' of the multisections locally, we have more freedom to make $\tilde{s}_i$ both transverse and Γ_i-equivariant. Then $\tilde{X}$ is not an orbifold, but a '$\mathbb{Q}$-weighted orbifold'. Done carefully, by triangulating by simplices we can define a virtual class $[\tilde{X}]$ in singular homology $H_{\dim\tilde{X}}(\tilde{X};\mathbb{Q})$, and then $[\mathbf{X}]_{\rm virt}=\tilde{f}_*([\tilde{X}])$.

It is important that although the construction of $[\mathbf{X}]_{\rm virt}$ involves many arbitrary choices of $(V_i,E_i,\Gamma_i,s_i,\psi_i),\tilde{s}_i,\ldots$, the final result $[\mathbf{X}]_{\rm virt}$ in $H_{\mathrm{vdim}\,\mathbf{X}}(Y;\mathbb{Q})$ is independent of these choices.

Furthermore, $[\mathbf{X}]_{\rm virt}$ is unchanged under bordisms of $\mathbf{f}:\mathbf{X}\to Y$, that is, it depends only on $[\mathbf{X},\mathbf{f}]\in dB^{\rm orb}_{\mathrm{vdim}\,\mathbf{X}}(Y)$. This bordism-independence makes Gromov–Witten invariants independent of the choice of almost complex structure J used to define them, so they are symplectic invariants, and so on.

12.3. Derived orbifolds with corners and virtual chains

In the Fukaya–Oh–Ono–Ohta Lagrangian Floer cohomology theory, given a symplectic manifold (M,ω) with an almost complex structure J and a Lagrangian L in M, one defines moduli spaces $\bar{\mathcal{M}}_k(\beta)$ of prestable J-holomorphic discs Σ in M with boundary in L, relative homology class $[\Sigma]=\beta\in H_2(M,L;\mathbb{Z})$, and k boundary marked points. The $\bar{\mathcal{M}}_k(\beta)$ are Kuranishi spaces with corners, with 'evaluation maps' $\mathbf{ev}_i:\bar{\mathcal{M}}_k(\beta)\to L$ for $i=1,\ldots,k$, and

$$\partial\bar{\mathcal{M}}_k(\beta)\simeq\coprod_{i+j=k}\coprod_{\beta_1+\beta_2=\beta}\bar{\mathcal{M}}_{i+1}(\beta_1)\times_{\mathbf{ev}_{i+1},L,\mathbf{ev}_{j+1}}\bar{\mathcal{M}}_{j+1}(\beta_2). \quad (12.1)$$

To define Lagrangian Floer cohomology, we have to 'count' the $\bar{\mathcal{M}}_k(\beta)$ in (co)homology, in some sense compatible with (12.1).

Roughly, we want to define a 'virtual class' for $\mathbf{ev}_1 \times \cdots \times \mathbf{ev}_k : \bar{\mathcal{M}}_k(\beta) \to L^k$ in $H_{\mathrm{vdim}\, \bar{\mathcal{M}}_k(\beta)}(L^k;\mathbb{Q})$. However, as $\partial\bar{\mathcal{M}}_k(\beta) \neq \emptyset$, we cannot define a homology class. Instead, we should write $H_*(L^k;\mathbb{Q})$ as the homology of a chain complex $\bigl(C_*(L^k;\mathbb{Q}),\partial\bigr)$, and define a *virtual chain* $[\bar{\mathcal{M}}_k(\beta)]_{\mathrm{virt}}$ in $C_{\mathrm{vdim}\, \bar{\mathcal{M}}_k(\beta)}(L^k;\mathbb{Q})$. The boundary $\partial C_{\mathrm{vdim}\, \bar{\mathcal{M}}_k(\beta)}(L^k;\mathbb{Q})$ should hopefully satisfy an equation modelled on (12.1), something like:

$$\partial[\bar{\mathcal{M}}_k(\beta)]_{\mathrm{virt}} = \sum_{i+j=k,\ \beta_1+\beta_2=\beta} [\bar{\mathcal{M}}_{i+1}(\beta_1)]_{\mathrm{virt}} \times_{\pi_{i+1},L,\pi_{j+1}} [\bar{\mathcal{M}}_{j+1}(\beta_2)]_{\mathrm{virt}}. \tag{12.2}$$

Fukaya–Oh–Ohta–Ono 2009 take their homology theory $\bigl(C_*(L^k;\mathbb{Q}),\partial\bigr)$ to be singular homology, generated by smooth maps $\sigma : \Delta_n \to L^k$. Then (12.2) does not make sense, as the fibre product (intersection product) is not defined in singular homology at the chain level. They have an alternative approach (unfinished (?), 2010) using de Rham cohomology, in which (12.2) does make sense.

There are other technical difficulties in the FOOO approach. One is that for each moduli space $\bar{\mathcal{M}}_k(\beta)$, one must choose a perturbation by multisections, and a triangulation by simplices / de Rham (co)chains, and these perturbations must be compatible at $\partial\bar{\mathcal{M}}_k(\beta)$ according to (12.1). There are infinitely many moduli spaces, and each moduli space $\bar{\mathcal{M}}_k(\beta)$ can occur in the formula for $\partial^j\bar{\mathcal{M}}_l(\beta')$ for infinitely many j,l,β', so the virtual chain for $\bar{\mathcal{M}}_k(\beta)$ should be subject to infinitely many compatibility conditions, including infinitely many smallness conditions on the size of the perturbations $s_i \rightsquigarrow \tilde{s}_i$. But we can only satisfy finitely many smallness conditions at once. And so on.

12.4. New (co)homology theories for virtual chains

A lot of the technical complexity in Fukaya–Oh–Ohta–Ono's 2009 Lagrangian Floer cohomology theory comes from the fact that the homology theory in which they do all their homological algebra – singular homology – does not play nicely with Kuranishi spaces. Their de Rham version is better, but still not ideal.

I would like to propose an alternative approach, which is to define new (co)homology theories $KH_*(Y;R), KH^*(Y;R)$ of manifolds Y, in which it is easy to define virtual classes and virtual chains for compact, oriented Kuranishi spaces $\mathbf{X}$ with 1-morphisms $\mathbf{f}:\mathbf{X}\rightarrow Y$. (Joyce, work in progress 2015; prototype version using FOOO Kuranishi spaces arXiv:0707.3573, arXiv:0710.5634 – please don't read these.) I'll discuss one version of the homology theory only.

Let Y be a manifold or orbifold, and R a $\mathbb{Q}$-algebra. We define a complex of R-modules $\bigl(KC_*(Y;R),\partial\bigr)$, whose homology groups $KH_*(Y;R)$ are the *Kuranishi homology* of Y.

Similarly to the definition of d-bordism $dB_k(Y)$, chains in $KC_k(Y;R)$ for $k\in\mathbb{Z}$ are R-linear combinations of equivalence classes $[\mathbf{X},\mathbf{f},\mathbf{G}]$ with relations, where $\mathbf{X}$ is a compact, oriented Kuranishi space with corners with dimension k, $\mathbf{f}:\mathbf{X}\rightarrow Y$ is a 1-morphism in $\mathbf{Kur^c}$, and $\mathbf{G}$ is some extra 'gauge-fixing data' associated to $\mathbf{f}:\mathbf{X}\rightarrow Y$, with many possible choices.

I won't give all the relations on the $[\mathbf{X},\mathbf{f},\mathbf{G}]$. Two examples are:

$$[\mathbf{X}_1\amalg\mathbf{X}_2,\mathbf{f}_1\amalg\mathbf{f}_2,\mathbf{G}_1\amalg\mathbf{G}_2]=[\mathbf{X}_1,\mathbf{f}_1,\mathbf{G}_1]+[\mathbf{X}_2,\mathbf{f}_2,\mathbf{G}_2], \tag{12.3}$$

$$[-\mathbf{X},\mathbf{f},\mathbf{G}]=-[\mathbf{X},\mathbf{f},\mathbf{G}], \tag{12.4}$$

where $-\mathbf{X}$ is $\mathbf{X}$ with the opposite orientation.

Gauge-fixing data – first properties

Here 'gauge-fixing data' is the key to the whole story. I won't tell you what it is, but I will tell you some properties it has:

(i) For any compact Kuranishi space with corners $\mathbf{X}$ and 1-morphism $\mathbf{f}:\mathbf{X}\to Y$ we have a nonempty set $\mathrm{Gauge}(\mathbf{f}:\mathbf{X}\to Y)$ of choices of 'gauge-fixing data' $\mathbf{G}$ for $\mathbf{f}$.

(ii) If $\mathbf{g}:\mathbf{X}'\to\mathbf{X}$ is étale we have a *pullback map* $\mathbf{g}^*:\mathrm{Gauge}(\mathbf{f}:\mathbf{X}\to Y)\to\mathrm{Gauge}(\mathbf{f}\circ\mathbf{g}:\mathbf{X}'\to Y)$. If $\mathbf{g},\mathbf{g}'$ are 2-isomorphic then $\mathbf{g}^*=\mathbf{g}'^*$. Pullbacks are functorial.

(iii) There is a *boundary map* $|_{\partial\mathbf{X}}:\mathrm{Gauge}(\mathbf{f}:\mathbf{X}\to Y)\to\mathrm{Gauge}(\mathbf{f}\circ\mathbf{i_X}:\partial\mathbf{X}\to Y)$. We regard it as a pullback along $\mathbf{i_X}:\partial\mathbf{X}\to\mathbf{X}$.

(iv) If $g:Y\to Z$ is a smooth map of manifolds, there is a *pushforward map* $g_*:\mathrm{Gauge}(\mathbf{f}:\mathbf{X}\to Y)\to\mathrm{Gauge}(g\circ\mathbf{f}:\mathbf{X}\to Z)$.

Boundary operators

Note that Kuranishi spaces $\mathbf{X}$ can have virtual dimension $\mathrm{vdim}\,\mathbf{X}<0$, so $KC_k(Y;R)\neq 0$ for all $k<0$, although $KH_k(Y;R)=0$ for $k<0$.
The boundary operator $\partial:KC_k(Y;R)\to KC_{k-1}(Y;R)$ maps

$$\partial:\bigl[\mathbf{X},\mathbf{f},\mathbf{G}\bigr]\longmapsto\bigl[\partial\mathbf{X},\mathbf{f}\circ\mathbf{i_X},\mathbf{G}|_{\partial\mathbf{X}}\bigr].$$

We have a natural 1-morphism $\mathbf{i_X}:\partial\mathbf{X}\to\mathbf{X}$ and an equivalence $\partial^2\mathbf{X}\simeq\partial\mathbf{X}_{\mathbf{i_X},\mathbf{X},\mathbf{i_X}}\partial\mathbf{X}$. Thus there is an orientation-reversing involution $\boldsymbol{\sigma}:\partial^2\mathbf{X}\to\partial^2\mathbf{X}$ swapping the two factors of $\partial\mathbf{X}$. This satisfies $\mathbf{i_X}\circ\mathbf{i}_{\partial\mathbf{X}}\circ\boldsymbol{\sigma}\cong\mathbf{i_X}\circ\mathbf{i}_{\partial\mathbf{X}}$. Hence $\mathbf{G}|_{\partial^2\mathbf{X}}$ is $\boldsymbol{\sigma}$-invariant. Using this and (12.4) we show that $\partial^2=0$, so $KH_*(Y;R)$ is well-defined.
Here is a property of gauge-fixing data with prescribed boundary values. 'Only if' is necessary by (i)–(iii) as above.

(v) Suppose $\mathbf{H}\in\mathrm{Gauge}(\mathbf{f}\circ\mathbf{i_X}:\partial\mathbf{X}\to Y)$. Then there exists $\mathbf{G}\in\mathrm{Gauge}(\mathbf{f}:\mathbf{X}\to Y)$ with $\mathbf{G}|_{\partial\mathbf{X}}=\mathbf{H}$ iff $\boldsymbol{\sigma}^*(\mathbf{H}|_{\partial^2\mathbf{X}})=\mathbf{H}|_{\partial^2\mathbf{X}}$.

Suppose $g : Y \to Z$ is a smooth map of manifolds or orbifolds. Define an R-linear *pushforward* $g_* : KC_k(Y;R) \to KC_k(Z;R)$ by $g_* : [\mathbf{X}, \mathbf{f}, \mathbf{G}] \mapsto [\mathbf{X}, g \circ \mathbf{f}, g_*(\mathbf{G})]$. Then $g_* \circ \partial = \partial \circ g_*$, so this induces $g_* : KH_k(Y;R) \to KH_k(Z;R)$. Pushforwards are functorial.

Singular homology $H_*^{\mathrm{sing}}(Y;R)$ may be defined using $\bigl(C_*^{\mathrm{sing}}(Y;R), \partial\bigr)$, where $C_k^{\mathrm{sing}}(Y;R)$ is spanned by *smooth* maps $f : \Delta_k \to Y$, for Δ_k the standard k-simplex, thought of as a manifold with corners.

We define an R-linear map $F_{\mathrm{sing}}^{\mathrm{KH}} : C_k^{\mathrm{sing}}(Y;R) \to KC_k(Y;R)$ by

$$F_{\mathrm{sing}}^{\mathrm{KH}} : f \longmapsto \bigl[\Delta_k, f, \mathbf{G}_{\Delta_k}\bigr],$$

with $\mathbf{G}_{\Delta_k}$ some standard choice of gauge-fixing data for Δ_k. Then $F_{\mathrm{sing}}^{\mathrm{KH}} \circ \partial = \partial \circ F_{\mathrm{sing}}^{\mathrm{KH}}$, so that $F_{\mathrm{sing}}^{\mathrm{KH}}$ induces morphisms $F_{\mathrm{sing}}^{\mathrm{KH}} : H_k^{\mathrm{sing}}(Y;R) \to KH_k(Y;R)$.

One of the main results of the theory will be:

Theorem 12.4

$F_{\mathrm{sing}}^{\mathrm{KH}} : H_k^{\mathrm{sing}}(Y;R) \to KH_k(Y;R)$ is an isomorphism for all $k \in \mathbb{Z}$.

- Forming virtual classes/virtual chains is easy. Suppose $\bar{\mathcal{M}}$ is a moduli Kuranishi space, with evaluation map $\mathbf{ev} : \bar{\mathcal{M}} \to Y$. Choose gauge-fixing data $\mathbf{G}$ for $\bar{\mathcal{M}}$, which is possible by (i). Then $[\bar{\mathcal{M}}, \mathbf{ev}, \mathbf{G}] \in KC_k(Y;R)$ is a virtual chain for $\bar{\mathcal{M}}$. If $\partial\bar{\mathcal{M}} = \emptyset$ then $[[\bar{\mathcal{M}}, \mathbf{ev}, \mathbf{G}]] \in KH_k(Y;R)$ is a virtual class for $\bar{\mathcal{M}}$.
- Obviously, Kuranishi homology is not a new invariant, it's just ordinary homology. The point is that it has special properties at the chain level which make it more convenient than competing homology theories (e.g. singular homology) for some tasks.
- The messy parts of defining virtual chains in [FOOO 2009] are repackaged in the proof of Theorem 12.4.

Derived Differential Geometry

Lecture 13 of 14: Existence of derived manifold or orbifold structures on moduli spaces

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Summer 2015

These slides, and references, etc., available at
http://people.maths.ox.ac.uk/∼joyce/DDG2015

Plan of talk:

13. Putting derived orbifold structures on moduli spaces

Suppose we have a moduli space $\mathcal{M}$ of some objects in differential geometry, or complex algebraic geometry, and we would like to make $\mathcal{M}$ into a derived manifold or derived orbifold (Kuranishi space) $\boldsymbol{\mathcal{M}}$, possibly with corners; either in order to form a virtual class/virtual chain for $\boldsymbol{\mathcal{M}}$ as in §12, or for some other reason. How do we go about this? There are two obvious methods:

(A) To somehow directly construct the derived orbifold $\boldsymbol{\mathcal{M}}$.

(B) Suppose we already know, e.g. by a theorem in the literature, that $\mathcal{M}$ carries some other geometric structure $\mathcal{G}$, such as a $\mathbb{C}$-scheme with perfect obstruction theory. Then we may be able to apply a 'truncation functor', a theorem saying that topological spaces X with geometric structure $\mathcal{G}$ can be made into derived manifolds or orbifolds $\mathbf{X}$.

Putting derived orbifold structures on moduli spaces
J-holomorphic curves and Gromov–Witten invariants
D-manifolds and nonlinear elliptic equations
Truncation functors from other structures
D-orbifolds as representable 2-functors
Moduli 2-functors in differential geometry

Which moduli problems give derived manifolds or orbifolds?

For a moduli space $\mathcal{M}$ of geometric objects E to form a derived manifold or orbifold $\boldsymbol{\mathcal{M}}$, roughly we need:

(a) Objects E should have at most finite symmetry groups (other than multiples of the identity in linear problems);

(b) Objects E can have deformations and obstructions, but no 'higher obstructions'; and

(c) Some global conditions on $\mathcal{M}$: Hausdorff, constant dimension.

In Differential Geometry, moduli spaces $\mathcal{M}$ of solutions of *nonlinear elliptic equations on compact manifolds* are almost automatically derived manifolds or orbifolds, as we explain in §13.1. This is a large class, which includes many important problems.

In Complex Algebraic Geometry, the deformation theory of objects E in $\mathcal{M}$ is usually understood either in terms of Ext groups $\mathrm{Ext}^i(E,E)$ for $i=0,1,\ldots,$ or sheaf cohomology groups $H^i(\Theta_E)$ of some sheaf Θ_E. Here $\mathrm{Ext}^0(E,E)$ or $H^0(\Theta_E)$ is the Lie algebra of the symmetry group of E; $\mathrm{Ext}^1(E,E)$ or $H^1(\Theta_E)$ the tangent space $T_E\boldsymbol{\mathcal{M}}$; $\mathrm{Ext}^2(E,E)$ or $H^2(\Theta_E)$ the obstruction space $O_E\boldsymbol{\mathcal{M}}$; and $\mathrm{Ext}^i(E,E)$ or $H^i(\Theta_E)$ for $i>2$ the 'higher obstruction spaces'. So to get a derived manifold or orbifold $\boldsymbol{\mathcal{M}}$, we need $\mathrm{Ext}^i(E,E)=0$ or $H^i(\Theta_E)=0$ for $i=0$ and $i>2$.
In linear problems we may restrict to the 'trace-free' part $\mathrm{Ext}^i(E,E)_0$. We get $\mathrm{Ext}^0(E,E)=0$ or $H^0(\Theta_E)=0$ by restricting to moduli spaces of 'stable' objects E.
$\mathrm{Ext}^i(E,E)=0$ or $H^i(\Theta_E)=0$ for $i>2$ may occur for dimensional reasons. It is automatic for E living on a curve or algebraic surface. For E on some classes of 3-folds (Calabi–Yau, Fano), we may have $\mathrm{Ext}^3(E,E)=0$ by Serre duality or vanishing theorems.

Briefly, the following classes of complex algebraic moduli spaces can usually be made into derived manifolds or orbifolds:

- Moduli spaces of Deligne–Mumford stable curves Σ in a smooth complex algebraic variety Y of any dimension.
- Moduli spaces of stable coherent sheaves / vector bundles / principal bundles on a Riemann surface, complex algebraic surface, Calabi–Yau 3-fold, Fano 3-fold, or Calabi–Yau 4-fold.

In Derived Algebraic Geometry, the main condition for a derived $\mathbb{C}$-stack $\mathbf{X}$ to be a derived manifold or orbifold is that it should be a locally finitely presented derived $\mathbb{C}$-scheme or Deligne–Mumford $\mathbb{C}$-stack which is *quasi-smooth*, i.e. has cotangent complex $\mathbb{L}_{\mathbf{X}}$ perfect in the interval $[-1,0]$.

13.1. D-manifolds and nonlinear elliptic equations

Elliptic equations are a class of p.d.e.s. They are determined (have the same number of equations as unknowns) and satisfy a nondegeneracy condition. Moduli problems with gauge symmetries are often elliptic after 'gauge-fixing'.

Elliptic equations are studied using functional analysis. For example, let Y be a compact manifold, $E, F \to Y$ be vector bundles, and $P : C^\infty(E) \to C^\infty(F)$ a linear partial differential operator of order k. For P to be elliptic we need $\mathrm{rank}\, E = \mathrm{rank}\, F$, and an invertibility condition on the k^{th} order derivatives in P.

Extend P to Hölder spaces $P : C^{k+l,\alpha}(E) \to C^{l,\alpha}(F)$ or Sobolev spaces $P : L^p_{k+l}(E) \to L^p_l(F)$. Then Y compact and P elliptic implies these maps are Fredholm maps between Banach spaces, with $\mathrm{Ker}\, P$, $\mathrm{Coker}\, P$ finite-dimensional, and the *index* $\mathrm{ind} P = \dim \mathrm{Ker}\, P - \dim \mathrm{Coker}\, P$ is given in terms of algebraic topology by the Atiyah–Singer Index Theorem.

Theorem 13.1

Let $\mathcal{V}$ be a Banach manifold, $\mathcal{E} \to \mathcal{V}$ a Banach vector bundle, and $s : \mathcal{V} \to \mathcal{E}$ a smooth Fredholm section, with constant Fredholm index $n \in \mathbb{Z}$. Then there is a d-manifold $\mathbf{X}$, unique up to equivalence in $\mathbf{dMan}$, with topological space $X = s^{-1}(0)$ and $\mathrm{vdim}\, \mathbf{X} = n$. If instead $\mathcal{V}$ is a Banach orbifold, or has boundary or corners, then the same thing holds with $\mathbf{X}$ a d-orbifold or Kuranishi space, or with boundary or corners.

Note that this basically says we can do 'standard model' d-manifolds $\mathbf{S}_{\mathcal{V},\mathcal{E},s}$ for (infinite-dimensional) Banach manifolds $\mathcal{V}$ and Banach vector bundles $\mathcal{E}$, with Fredholm sections s.

To prove Theorem 13.1, near each $x \in s^{-1}(0)$ we use the Implicit Function Theorem for Banach spaces and Fredholmness to show $s^{-1}(0)$ is locally modelled on $\tilde{s}^{-1}(0)$ for $\tilde{V}$ a manifold, $\tilde{E} \to \tilde{V}$ a vector bundle, and $\tilde{s} \in C^\infty(\tilde{E})$. Then we combine these Kuranishi neighbourhoods $(\tilde{V}, \tilde{E}, \tilde{s})$ into a d-manifold/Kuranishi structure on X.

Nonlinear elliptic equations, when written as maps between suitable Hölder or Sobolev spaces, become the zeroes $s = 0$ of Fredholm sections s of a (possibly trivial) Banach vector bundle $\mathcal{E} \to \mathcal{V}$ over a Banach manifold (or Banach space) $\mathcal{V}$. Thus we have:

Corollary 13.2

Let $\mathcal{M}$ be a moduli space of solutions of a nonlinear elliptic equation on a compact manifold, with fixed topological invariants. Then $\mathcal{M}$ extends to a d-manifold $\boldsymbol{\mathcal{M}}$.

The virtual dimension $\boldsymbol{\mathcal{M}}$ at $x \in \boldsymbol{\mathcal{M}}$ is the index of the (Fredholm) linearization of the nonlinear elliptic equation at x, which is given by the A–S Index Theorem. We require *fixed topological invariants* so this dimension is constant over $\boldsymbol{\mathcal{M}}$.
Note that Corollary 13.2 does *not* include problems involving dividing by a gauge group, since such gauge groups typically act only continuously on the Banach manifold. Nonetheless, a similar result should hold for nonlinear elliptic equations modulo gauge.

Example 13.3

Let (X, g), (Y, h) be Riemannian manifolds, with X compact. The moduli space $\mathcal{M}$ of *harmonic maps* $f : X \to Y$ is defined by a nonlinear elliptic equation, and so becomes a d-manifold $\boldsymbol{\mathcal{M}}$, with $\mathrm{vdim}\, \boldsymbol{\mathcal{M}} = 0$. For instance, when $X = \mathcal{S}^1$, $\boldsymbol{\mathcal{M}}$ is the moduli space of parametrized closed geodesics in (Y, h).

Example 13.4

Let (Σ, j) be a Riemann surface, and (Y, J) a manifold with almost complex structure. Then the moduli space $\mathcal{M}(\beta)$ of (j, J)-holomorphic maps $u : \Sigma \to Y$ with $u_*([\Sigma]) = \beta \in H_2(Y; \mathbb{Z})$ is defined by an elliptic equation, and is a a d-manifold $\boldsymbol{\mathcal{M}}(\beta)$.
Note that (Σ, j) is a *fixed*, *nonsingular* Riemann surfaces. Moduli spaces in which (Σ, j) is allowed to vary (and especially, allowed to become singular) are more complicated.

13.2. Truncation functors from other structures
Fukaya–Oh–Ohta–Ono Kuranishi spaces

Fukaya–Ono 1999 and Fukaya–Oh–Ohta–Ono 2009 defined their version of Kuranishi spaces, which we call *FOOO Kuranishi spaces*.

Theorem 13.5

Let $\mathbf{X}$ *be a FOOO Kuranishi space. Then we can define a Kuranishi space* $\mathbf{X}'$ *in the sense of* §8, *canonical up to equivalence in the* 2*-category* **Kur**, *with the same topological space as* $\mathbf{X}$.
The same holds for other Kuranishi-space-like structures in the literature, such as McDuff–Wehrheim's 'Kuranishi atlases', 2012.

Therefore any moduli space which has been proved to carry a FOOO Kuranishi space structure (many J-holomorphic curve moduli spaces) is also a Kuranishi space/d-orbifold in our sense. FOOO Kuranishi spaces do not form a category, so Theorem 13.5 does not give a 'truncation functor'.

Hofer–Wysocki–Zehnder's polyfolds

Polyfolds, due to Hofer, Wysocki and Zehnder (2005–2015+), are a rival theory to FOOO Kuranishi spaces. They do form a category. Polyfolds remember much more information than Kuranishi spaces.

Theorem 13.6

There is a functor $\Pi_{\mathbf{PolFS}}^{\mathbf{dOrb^c}}:\mathbf{PolFS}\rightarrow \mathrm{Ho}(\mathbf{Kur})$, *where* **PolFS** *is a category whose objects are triples* $(\mathcal{V},\mathcal{E},s)$ *of a polyfold* $\mathcal{V}$, *a fillable strong polyfold bundle* $\mathcal{E}$ *over* $\mathcal{V}$, *and an sc-smooth Fredholm section* s *of* $\mathcal{E}$ *with constant Fredholm index.*

Here $\mathrm{Ho}(\mathbf{Kur})$ is the homotopy category of the 2-category **Kur**. Combining the theorem with constructions of polyfold structures on moduli spaces (e.g. HWZ arXiv:1107.2097, J-holomorphic curves for G–W invariants), gives d-orbifold structures on moduli spaces.

$\mathbb{C}$-schemes and $\mathbb{C}$-stacks with obstruction theories

In algebraic geometry, the standard method of forming virtual cycles is to use a proper scheme or Deligne–Mumford stack equipped with a *perfect obstruction theory* (Behrend–Fantechi). They are used to define algebraic Gromov–Witten invariants, Donaldson–Thomas invariants of Calabi–Yau 3-folds, ...

Theorem 13.7

There is a functor $\Pi^{\mathbf{dMan}}_{\mathbf{SchObs}} : \mathbf{Sch}_{\mathbb{C}}\mathbf{Obs} \to \mathrm{Ho}(\mathbf{dMan})$, *where* $\mathbf{Sch}_{\mathbb{C}}\mathbf{Obs}$ *is a category whose objects are triples* $(X, E^\bullet, \phi)$, *for* X *a separated, second countable* $\mathbb{C}$*-scheme and* $\phi : E^\bullet \to \mathbb{L}_X$ *a perfect obstruction theory on* X *with constant virtual dimension.*
The analogue holds for $\Pi^{\mathbf{dOrb}}_{\mathbf{StaObs}} : \mathbf{Sta}_{\mathbb{C}}\mathbf{Obs} \to \mathrm{Ho}(\mathbf{dOrb})$, *replacing* $\mathbb{C}$*-schemes by Deligne–Mumford* $\mathbb{C}$*-stacks, and d-manifolds by d-orbifolds (or equivalently Kuranishi spaces, using* $\mathrm{Ho}(\mathbf{Kur})$*).*

So, many $\mathbb{C}$-algebraic moduli spaces are d-manifolds or d-orbifolds.

Derived $\mathbb{C}$-schemes and Deligne–Mumford $\mathbb{C}$-stacks

Theorem 13.8

There is a functor $\Pi^{\mathbf{dMan}}_{\mathbf{dSch}} : \mathrm{Ho}(\mathbf{dSch}^{\mathbf{qs}}_{\mathbb{C}}) \to \mathrm{Ho}(\mathbf{dMan})$, *where* $\mathrm{Ho}(\mathbf{dSch}^{\mathbf{qs}}_{\mathbb{C}})$ *is the homotopy category of the* ∞*-category of derived* $\mathbb{C}$*-schemes* $\mathbf{X}$, *where* $\mathbf{X}$ *is assumed locally finitely presented, separated, second countable, of constant virtual dimension, and* **quasi-smooth**, *that is,* $\mathbb{L}_{\mathbf{X}}$ *is perfect in the interval* $[-1, 0]$.
The analogue holds for $\Pi^{\mathbf{dOrb}}_{\mathbf{dSta}} : \mathrm{Ho}(\mathbf{dSta}^{\mathbf{qs}}_{\mathbb{C}}) \to \mathrm{Ho}(\mathbf{dOrb})$, *replacing derived* $\mathbb{C}$*-schemes by derived Deligne–Mumford* $\mathbb{C}$*-stacks, and d-manifolds by d-orbifolds (or Kuranishi spaces).*

Actually this follows from Theorem 13.7, since if $\mathbf{X}$ is a quasi-smooth derived $\mathbb{C}$-scheme then the classical truncation $X = t_0(\mathbf{X})$ is a $\mathbb{C}$-scheme with perfect obstruction theory $\mathbb{L}_i : \mathbb{L}_{\mathbf{X}}|_X \to \mathbb{L}_X$, for $i : X \hookrightarrow \mathbf{X}$ the inclusion.

−2-shifted symplectic derived $\mathbb{C}$-schemes

Theorem 13.9 (Borisov–Joyce arXiv:1504.00690)

Suppose $\boldsymbol{X}$ *is a derived* $\mathbb{C}$*-scheme with a* -2*-shifted symplectic structure* $\omega_{\boldsymbol{X}}$ *in the sense of Pantev–Toën–Vaquié–Vezzosi* arXiv:1111.3209. *Then we can define a d-manifold* $\boldsymbol{X}_{\rm dm}$ *with the same underlying topological space, and virtual dimension* $\mathrm{vdim}_{\mathbb{R}}\boldsymbol{X}_{\rm dm}=\frac{1}{2}\mathrm{vdim}_{\mathbb{R}}\boldsymbol{X}$, *i.e. half the expected dimension.*

Note that $\boldsymbol{X}$ is not quasi-smooth, $\mathbb{L}_{\boldsymbol{X}}$ lies in the interval $[-2,0]$, so this does not follow from Theorem 13.8. Also $\boldsymbol{X}_{\rm dm}$ is only canonical up to bordisms fixing the underlying topological space.
Derived moduli schemes or stacks of coherent sheaves on a Calabi–Yau m-fold are $(2-m)$-shifted symplectic, so this gives:

Corollary 13.10

Stable moduli schemes of coherent sheaves $\mathcal{M}$ *with fixed Chern character on a Calabi–Yau* 4*-fold can be made into d-manifolds* $\boldsymbol{\mathcal{M}}$.

13.3. D-orbifolds as representable 2-functors

Disclaimer: the rest of this lecture is work in progress (or more honestly, not yet begun). I'm fairly confident it will work eventually.

Recall the Grothendieck approach to moduli spaces in algebraic geometry from §1.3, using *moduli functors*. Write $\mathbf{Sch}_{\mathbb{C}}$ for the category of $\mathbb{C}$-schemes, and $\mathbf{Sch}^{\mathbf{aff}}_{\mathbb{C}}$ for the subcategory of affine $\mathbb{C}$-schemes. Any $\mathbb{C}$-scheme X defines a functor $\mathrm{Hom}(-,X):\mathbf{Sch}^{\mathbf{op}}_{\mathbb{C}}\to\mathbf{Sets}$ mapping each $\mathbb{C}$-scheme S to the set $\mathrm{Hom}(S,X)$, where $\mathbf{Sch}^{\mathbf{op}}_{\mathbb{C}}$ is the *opposite category* to $\mathbf{Sch}_{\mathbb{C}}$ (reverse directions of morphisms). By the Yoneda Lemma, the $\mathbb{C}$-scheme X is determined up to isomorphism by the functor $\mathrm{Hom}(-,X)$ up to natural isomorphism. This is still true if we restrict to $\mathbf{Sch}^{\mathbf{aff}}_{\mathbb{C}}$. Thus, given a functor $F:(\mathbf{Sch}^{\mathbf{aff}}_{\mathbb{C}})^{\mathbf{op}}\to\mathbf{Sets}$, we can ask if there exists a $\mathbb{C}$-scheme X (necessarily unique up to canonical isomorphism) with $F\cong\mathrm{Hom}(-,X)$. If so, we call F a *representable functor*.

Classical stacks

As in §1.4, to extend this from $\mathbb{C}$-schemes to Deligne–Mumford or Artin $\mathbb{C}$-stacks, we consider functors $F : (\mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}})^{\mathbf{op}} \rightarrow \mathbf{Groupoids}$, where a groupoid is a category all of whose morphisms are isomorphisms. (We can regard a set as a category all of whose morphisms are identities, so replacing **Sets** by **Groupoids** is a generalization.)

A *stack* is a functor $F : (\mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}})^{\mathbf{op}} \rightarrow \mathbf{Groupoids}$ satisfying a sheaf-type condition: if S is an affine $\mathbb{C}$-scheme and $\{S_i : i \in I\}$ an open cover of S (in some algebraic topology) then we should be able to reconstruct $F(S)$ from $F(S_i)$, $F(S_i \cap S_j)$, $F(S_i \cap S_j \cap S_k)$, $i, j, k \in I$, and the functors between them.
A Deligne–Mumford or Artin $\mathbb{C}$-stack is a stack $F : (\mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}})^{\mathbf{op}} \rightarrow \mathbf{Groupoids}$ satisfying extra geometric conditions.

Grothendieck's moduli schemes

Suppose we have an algebro-geometric moduli problem (e.g. vector bundles on a smooth projective $\mathbb{C}$-scheme Y) for which we want to form a moduli scheme. Grothendieck tells us that we should define a *moduli functor* $F : (\mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}})^{\mathbf{op}} \rightarrow \mathbf{Sets}$, such that for each affine $\mathbb{C}$-scheme S, $F(S)$ is the set of isomorphism classes of families of the relevant objects over S (e.g. vector bundles over $Y \times S$). Then we should try to prove F is a representable functor, using some criteria for representability. If it is, $F \cong \mathrm{Hom}(-, \mathcal{M})$, where $\mathcal{M}$ is the (*fine*) *moduli scheme*.

To form a *moduli stack*, we define $F : (\mathbf{Sch}_{\mathbb{C}}^{\mathbf{aff}})^{\mathbf{op}} \rightarrow \mathbf{Groupoids}$, so that for each affine $\mathbb{C}$-scheme S, $F(S)$ is the groupoid of families of objects over S, with morphisms isomorphisms of families, and try to show F satisfies the criteria to be an Artin stack.

D-orbifolds as representable 2-functors

D-orbifolds $\mathbf{dOrb}$ (or Kuranishi spaces $\mathbf{Kur}$) are a 2-category with all 2-morphisms invertible. Thus, if $\mathbf{S}, \mathbf{X} \in \mathbf{dOrb}$ then $\mathbf{Hom}(\mathbf{S}, \mathbf{X})$ is a groupoid, and $\mathbf{Hom}(-, \mathbf{X}) : \mathbf{dOrb}^{\mathbf{op}} \to \mathbf{Groupoids}$ is a 2-functor, which determines $\mathbf{X}$ up to equivalence in $\mathbf{dOrb}$. This is still true if we restrict to affine (meaning standard model) d-manifolds $\mathbf{dMan}^{\mathbf{aff}} \subset \mathbf{dOrb}$. Thus, we can consider 2-functors $F : (\mathbf{dMan}^{\mathbf{aff}})^{\mathbf{op}} \to \mathbf{Groupoids}$, and ask whether there exists a d-orbifold $\mathbf{X}$ (unique up to equivalence) with $F \simeq \mathbf{Hom}(-, \mathbf{X})$. If so, we call F a *representable 2-functor*.

Why use $(\mathbf{dMan}^{\mathbf{aff}})^{\mathbf{op}}$ as the domain of the functor? A d-orbifold $\mathbf{X}$ also induces a functors $\mathrm{Hom}(-, \mathbf{X}) : \mathcal{C}^{\mathbf{op}} \to \mathbf{Groupoids}$ for $\mathcal{C} = \mathbf{Man}, \mathbf{Orb}, \mathbf{C^\infty Sch}, \mathbf{C^\infty Sta}, \mathbf{dMan}, \mathbf{dOrb}, \mathbf{dSpa}, \mathbf{dSta}, \ldots$. We want $\mathcal{C}$ large enough that $\mathbf{dOrb} \hookrightarrow \mathbf{Funct}(\mathcal{C}^{\mathbf{op}}, \mathbf{Groupoids})$ is an embedding, but otherwise as small as possible, as we must prove things for all objects in $\mathcal{C}$, so a smaller $\mathcal{C}$ saves work.

Criteria for representable 2-functors

Let $F : (\mathbf{dMan}^{\mathbf{aff}})^{\mathbf{op}} \to \mathbf{Groupoids}$ be a functor. When is F representable (that is, $F \simeq \mathrm{Hom}(-, \mathbf{X})$ for some d-orbifold $\mathbf{X}$)? It is good to have usable *criteria for representability*, such that if one can show the criteria hold in an example, then we know F is representable (even without constructing the d-orbifold $\mathbf{X}$).

I expect there are nice criteria of the form:

(A) F satisfies a sheaf-type condition, i.e. F is a *stack*;

(B) the 'coarse topological space' $\mathcal{M} = F(\text{point})/\text{isos of } F$ is Hausdorff and second countable, and each point x of $\mathcal{M}$ has finite stabilizer group $\mathrm{Aut}(x)$; and

(C) F admits a 'Kuranishi neighbourhood' of dimension $n \in \mathbb{Z}$ near each $x \in \mathcal{M}$, a local model with a universal property.

Functors satisfying (A) (*stacks*) are a kind of geometric space, even if they are not d-orbifolds. They have points, and a topology, and one can work locally on them.

13.4. Moduli 2-functors in differential geometry

Suppose we are given a moduli problem in differential geometry (e.g. J-holomorphic curves in a symplectic manifold) and we want to form a moduli space $\mathcal{M}$ as a d-orbifold. I propose that we should define a *moduli 2-functor* $F : (\mathbf{dMan^{aff}})^{\mathbf{op}} \to \mathbf{Groupoids}$, such that for each affine d-manifold $\mathbf{S}$, $F(\mathbf{S})$ is the category of families of the relevant objects over $\mathbf{S}$. Then we should try to prove F satisfies (A)–(C), and so is represented by a d-orbifold $\mathcal{M}$; here (A),(B) will usually be easy, and (C) the difficult part.

If F is represented by $\mathcal{M}$, then there will automatically exist a *universal family* of objects over $\mathcal{M}$.

Example: moduli functors of J-holomorphic curves

Let (M,ω) be a symplectic manifold, and J an almost complex structure on M. Suppose we want to construct $F : (\mathbf{dMan^{aff}})^{\mathbf{op}} \to \mathbf{Groupoids}$ representing the moduli space of J-holomorphic maps $u : \Sigma \to M$, where (Σ, j) is a nonsingular genus g Riemann surface, and $[u(\Sigma)] = \beta \in H_2(M;\mathbb{Z})$.
Then, for each affine d-manifold $\mathbf{S}$, we must construct a groupoid $F(\mathbf{S})$ of families of J-holomorphic maps $u : \Sigma \to M$ over the base $\mathbf{S}$. There is a natural way to do this:

- Objects of $F(\mathbf{S})$ are quadruples $(\mathbf{X}, \boldsymbol{\pi}, \mathbf{u}, j)$, where $\mathbf{X}$ is a d-manifold with $\mathrm{vdim}\,\mathbf{X} = \mathrm{vdim}\,\mathbf{S} + 2$, $\boldsymbol{\pi} : \mathbf{X} \to \mathbf{S}$ a proper submersion of d-manifolds with $\boldsymbol{\pi}^{-1}(s)$ a genus g surface for all $s \in \mathbf{S}$, $\mathbf{u} : \mathbf{X} \to M$ is a 1-morphism with $[\mathbf{u}(\boldsymbol{\pi}^{-1}(S))] = \beta$ for all $s \in \mathbf{S}$, and $j : \mathbb{T}_{\boldsymbol{\pi}} \to \mathbb{T}_{\boldsymbol{\pi}}$ is bundle linear with $j^2 = -\mathrm{id}$ and $\mathbf{u}^*(J)\circ\mathrm{d}\mathbf{u} = \mathrm{d}\mathbf{u}\circ j$, for $\mathbb{T}_{\boldsymbol{\pi}}$ the relative tangent bundle of $\boldsymbol{\pi}$.

- Morphisms $[\mathbf{i},\eta,\zeta]:(\mathbf{X},\boldsymbol{\pi},\mathbf{u},j)\to(\mathbf{X}',\boldsymbol{\pi}',\mathbf{u}',j')$ in $F(\mathbf{S})$ are $\sim$-equivalence classes $[\mathbf{i},\eta,\zeta]$ of triples $(\mathbf{i},\eta,\zeta)$, where $\mathbf{i}:\mathbf{X}\to\mathbf{X}'$ is an equivalence in **dMan**, and $\eta:\boldsymbol{\pi}\Rightarrow\boldsymbol{\pi}'\circ\mathbf{i}$, $\zeta:\mathbf{u}\Rightarrow\mathbf{u}'\circ\mathbf{i}$ are 2-morphisms, and $H^0(\mathrm{d}\mathbf{i})$ identifies j,j', and $(\mathbf{i},\eta,\zeta)\sim(\tilde{\boldsymbol{\imath}},\tilde\eta,\tilde\zeta)$ if there exists a 2-morphism $\alpha:\mathbf{i}\Rightarrow\tilde{\boldsymbol{\imath}}$ with $\tilde\eta=(\mathrm{id}_{\boldsymbol{\pi}'}*\alpha)\odot\eta$ and $\tilde\zeta=(\mathrm{id}_{\mathbf{u}'}*\alpha)\odot\zeta$.
- If $\mathbf{f}:\mathbf{T}\to\mathbf{S}$ is a 1-morphism in $\mathbf{dMan^{aff}}$, the functor $F(\mathbf{f}):F(\mathbf{S})\to F(\mathbf{T})$ acts by $F(\mathbf{f}):(\mathbf{X},\boldsymbol{\pi},\mathbf{u},j)\mapsto(\mathbf{X}\times_{\boldsymbol{\pi},\mathbf{S},\mathbf{f}}\mathbf{T},\boldsymbol{\pi}_{\mathbf{T}},\mathbf{u}\circ\boldsymbol{\pi}_{\mathbf{X}},\boldsymbol{\pi}_{\mathbf{X}}^*(j))$ on objects and in a natural way on morphisms, with $\mathbf{X}\times_{\boldsymbol{\pi},\mathbf{S},\mathbf{f}}\mathbf{T}$ the fibre product in **dMan**.
- If $\mathbf{f},\mathbf{g}:\mathbf{T}\to\mathbf{S}$ are 1-morphisms and $\theta:\mathbf{f}\Rightarrow\mathbf{g}$ a 2-morphism in $\mathbf{dMan^{aff}}$, then $F(\theta):F(\mathbf{f})\Rightarrow F(\mathbf{g})$ is a natural isomorphism of functors, $F(\theta):(\mathbf{X},\boldsymbol{\pi},\mathbf{u},j)\mapsto[\mathbf{i},\eta,\zeta]$ for $(\mathbf{X},\boldsymbol{\pi},\mathbf{u},j)$ in $F(\mathbf{S})$, where $[\mathbf{i},\eta,\zeta]:(\mathbf{X}\times_{\boldsymbol{\pi},\mathbf{S},\mathbf{f}}\mathbf{T},\boldsymbol{\pi}_{\mathbf{T}},\mathbf{u}\circ\boldsymbol{\pi}_{\mathbf{X}},\boldsymbol{\pi}_{\mathbf{X}}^*(j))\to(\mathbf{X}\times_{\boldsymbol{\pi},\mathbf{S},\mathbf{g}}\mathbf{T},\boldsymbol{\pi}_{\mathbf{T}},\mathbf{u}\circ\boldsymbol{\pi}_{\mathbf{X}},\boldsymbol{\pi}_{\mathbf{X}}^*(j))$ in $F(\mathbf{T})$, with $\mathbf{i}:\mathbf{X}\times_{\boldsymbol{\pi},\mathbf{S},\mathbf{f}}\mathbf{T}\to\mathbf{X}\times_{\boldsymbol{\pi},\mathbf{S},\mathbf{g}}\mathbf{T}$ induced by $\theta:\mathbf{f}\Rightarrow\mathbf{g}$.

Conjecture 13.11

The moduli functor $F:(\mathbf{dMan^{aff}})^{\mathbf{op}}\to\mathbf{Groupoids}$ *above is represented by a d-orbifold.*

Some remarks:

- I may have got the treatment of almost complex structures in the definition of F wrong — this is a first guess.
- I expect to be able to prove Conjecture 13.11 (perhaps after correcting the definition). The proof won't be specific to J-holomorphic curves — there should be a standard method for proving representability of moduli functors of solutions of nonlinear elliptic equations with gauge symmetries, which would also work for many other classes of moduli problems.
- Proving Conjecture 13.11 will involve verifying the representability criteria (A)–(C) above for F.

- The definition of F involves fibre products $\mathbf{X} \times_{\pi,\mathbf{S},\mathbf{f}} \mathbf{T}$ in **dMan**, which exist as $\pi : \mathbf{X} \to \mathbf{S}$ is a submersion. Existence of suitable fibre products is *crucial* for the representable 2-functor approach. This becomes complicated when boundaries and corners are involved – see §11.
- Current definitions of differential-geometric moduli spaces (e.g. Kuranishi spaces, polyfolds) are generally very long, complicated ad hoc constructions, with no obvious naturality. In contrast, if we allow differential geometry over d-manifolds, my approach gives you a short, natural definition of the moduli functor F (only 2 slides above give a nearly complete definition!), followed by a long proof that F is representable. The effort moves from a construction to a theorem.
- Can write $\mathbf{X}, \mathbf{S}$ as 'standard model' d-manifolds, as in §5, and $\boldsymbol{\pi}, \mathbf{f}, \eta, \zeta, \ldots$ as 'standard model' 1- and 2-morphisms. Thus, can express F in terms of Kuranishi neighbourhoods and classical differential geometry.

- The definition of F involves *only finite-dimensional families of smooth objects*, with no analysis, Banach spaces, etc. (But the proof of (C) will involve analysis and Banach spaces.) This enables us to sidestep some analytic problems.
- In some problems, there will be several moduli spaces, with morphisms between them. E.g. if we include *marked points* in our J-holomorphic curves (do this by modifying objects $(\mathbf{X}, \boldsymbol{\pi}, \mathbf{u}, j)$ in $F(\mathbf{S})$ to include morphisms $\mathbf{z}_1, \ldots, \mathbf{z}_k : \mathbf{S} \to \mathbf{X}$ with $\boldsymbol{\pi} \circ \mathbf{z}_i \simeq \mathbf{id}_{\mathbf{S}}$), then we can have 'forgetful functors' between moduli spaces forgetting some of the marked points. Such forgetful functors appear as 2-*natural transformations* $\Theta : F \Rightarrow G$ between moduli functors $F, G : (\mathbf{dMan^{aff}})^{\mathbf{op}} \to \mathbf{Groupoids}$. If F, G are representable, forgetful functors induce 1-morphisms between the d-orbifolds.

Derived Differential Geometry

Lecture 14 of 14: J-holomorphic curves and Gromov–Witten invariants

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Summer 2015

These slides, and references, etc., available at
`http://people.maths.ox.ac.uk/~joyce/DDG2015`

Plan of talk:

14. J-holomorphic curves and Gromov–Witten invariants

14.1. J-holomorphic curves

An *almost complex structure* J on a $2n$-manifold S is a tensor J_a^b on S with $J_a^b J_b^c = -\delta_a^c$. For $v \in C^\infty(TS)$ define $(Jv)^b = J_a^b v^a$. Then $J^2 = -1$, so J makes the tangent spaces T_pS into complex vector spaces. If J is integrable then (S, J) is a complex manifold.
Now let (S, ω) be a symplectic manifold. An almost complex structure J on S is *compatible with* ω if $g = g_{ab} = \omega_{ac} J_b^c$ is symmetric and positive definite (i.e. a Riemannian metric) on S. If J is integrable then (S, J, g, ω) is Kähler.
Every symplectic manifold (S, ω) admits compatible almost complex structures J, and the (infinite-dimensional) family of such almost complex structures is contractible. So, in particular, given J_0, J_1, there exists a smooth family $J_t : t \in [0, 1]$ of compatible almost complex structures on (S, ω) interpolating between J_0 and J_1.

Let (S, ω) be symplectic, with almost complex structure J. A *pseudoholomorphic curve* or *J-holomorphic curve* in S is a Riemann surface (Σ, j) (almost always compact, sometimes singular) with a smooth map $u : \Sigma \to S$ such that $J \circ \mathrm{d}u = \mathrm{d}u \circ j : T\Sigma \to u^*(TS)$.
Moduli spaces of J-holomorphic curves $\bar{\mathcal{M}}$ in S behave a lot like moduli spaces of curves in complex manifolds, or smooth complex varieties; they do not really care that J is not integrable.
The importance of the symplectic structure is that

$$\mathrm{Area}_g(u(\Sigma)) = \int_{u(\Sigma)} \omega = [\omega] \cdot u_*([\Sigma]),$$

where $u_*([\Sigma]) \in H_2(S; \mathbb{Z})$ and $[\omega] \in H^2_{\mathrm{dR}}(S; \mathbb{R})$, and the area is computed using $g_{ab} = \omega_{ac} J_b^c$. Thus, J-holomorphic curves $u : \Sigma \to S$ in a fixed homology class in $H_2(S; \mathbb{Z})$ have a fixed, and hence bounded, area in S. This helps to ensure moduli spaces $\bar{\mathcal{M}}$ of J-holomorphic curves are compact (as areas of curves cannot go to infinity at noncompact ends of $\bar{\mathcal{M}}$), which is crucial.

Several important areas of symplectic geometry — Gromov–Witten invariants, Lagrangian Floer cohomology, Fukaya categories, contact homology, Symplectic Field Theory, ... — work as follows:

(a) Given a symplectic manifold (S,ω) (etc.), choose compatible J and define moduli spaces $\bar{\mathcal{M}}$ of J-holomorphic curves in S.
(b) Show $\bar{\mathcal{M}}$ is a compact, oriented Kuranishi space (or similar), possibly with corners.
(c) Form a virtual class / virtual chain $[\bar{\mathcal{M}}]_{\rm virt}$ for $\bar{\mathcal{M}}$.
(d) Do homological algebra with these $[\bar{\mathcal{M}}]_{\rm virt}$ to define Gromov–Witten invariants, Lagrangian Floer cohomology, etc.
(e) Prove the results are independent of the choice of J (up to isomorphism), so depend only on (S,ω) (etc.).
(f) Use the machine you have created to prove interesting stuff about symplectic manifolds, Lagrangian submanifolds, ...

We will explain Gromov–Witten invariants.

J-holomorphic curves with marked points

Let (S,ω) be a symplectic manifold, and J an almost complex structure on S compatible with ω. The obvious way to define moduli spaces of J-holomorphic curves is as sets of isomorphism classes $[\Sigma,u]$ of pairs (Σ,u), where Σ is a Riemann surface, and $u:\Sigma\rightarrow S$ is J-holomorphic.

But we will do something more complicated. We consider moduli spaces of J-holomorphic curves *with marked points*.

Our moduli spaces $\bar{\mathcal{M}}_{g,m}(S,J,\beta)$ will be sets of isomorphism classes $[\Sigma,\vec z,u]$ of triples $(\Sigma,\vec z,u)$, where Σ is a Riemann surface, $\vec z=(z_1,\ldots,z_m)$ with $z_1,\ldots,z_m$ points of Σ called *marked points*, and $u:\Sigma\rightarrow S$ is J-holomorphic. The point of this is that we then have *evaluation maps* ${\rm ev}_i:\bar{\mathcal{M}}_{g,m}(S,J,\beta)\rightarrow S$ for $i=1,\ldots,m$ mapping ${\rm ev}_i:[\Sigma,\vec z,u]\mapsto u(z_i)$.

Moduli spaces of nonsingular curves

We first discuss moduli spaces of Riemann surfaces without maps to a symplectic manifold. Fix $g, m \geqslant 0$. Consider pairs $(\Sigma, \vec{z})$, where Σ is a compact, nonsingular Riemann surface with genus g, and $\vec{z} = (z_1, \ldots, z_m)$ are distinct points of Σ. An *isomorphism* between $(\Sigma, \vec{z})$ and $(\Sigma', \vec{z}')$ is a biholomorphism $f : \Sigma \to \Sigma'$ with $f(z_i) = z_i'$ for $i = 1, \ldots, m$. Write $[\Sigma, \vec{z}]$ for the *isomorphism class* of $(\Sigma, \vec{z})$, that is, the equivalence class of $(\Sigma, \vec{z})$ under the equivalence relation of isomorphism.

The *automorphism group* $\mathrm{Aut}(\Sigma, \vec{z})$ is the group of automorphisms f from $(\Sigma, \vec{z})$ to $(\Sigma, \vec{z})$. We call $(\Sigma, \vec{z})$ *stable* if $\mathrm{Aut}(\Sigma, \vec{z})$ is finite. Otherwise $(\Sigma, \vec{z})$ is *unstable*. In fact $(\Sigma, \vec{z})$ is stable iff $g = 0$ and $m \geqslant 3$, or $g = 1$ and $m \geqslant 1$, or $g \geqslant 2$, that is, if $2g + m \geqslant 3$. But if we allow *singular* Σ then there can be unstable $(\Sigma, \vec{z})$ for any g, m. We will exclude unstable $(\Sigma, \vec{z})$ as they would make our moduli spaces non-Hausdorff.

Define $\mathcal{M}_{g,m}$ to be the set of isomorphism classes $[\Sigma, \vec{z}]$ of stable pairs $(\Sigma, \vec{z})$ with Σ nonsingular of genus g and m marked points $\vec{z} = (z_1, \ldots, z_m)$. By studying the deformation theory of pairs $(\Sigma, \vec{z})$ one can prove:

Theorem

$\mathcal{M}_{g,m}$ has the structure of a complex orbifold of complex dimension $3g + m - 3$. It is Hausdorff, but noncompact in general.

Here a *complex orbifold* M is a complex manifold with only quotient singularities. That is, M is locally modelled on $\mathbb{C}^n/\Gamma$ for Γ a finite group acting linearly on $\mathbb{C}^n$. The orbifold singularities of $\mathcal{M}_{g,m}$ come from $[\Sigma, \vec{z}]$ with $\mathrm{Aut}(\Sigma, \vec{z})$ nontrivial; $\mathcal{M}_{g,m}$ is locally modelled near $[\Sigma, \vec{z}]$ on $\mathbb{C}^{3g+m-3}/\mathrm{Aut}(\Sigma, \vec{z})$.

In Gromov–Witten theory, we must work with orbifolds rather than manifolds. This means that G–W invariants are *rational numbers* rather than integers, since the ‘number of points’ in the 0-orbifold $\{0\}/\Gamma$ should be $1/|\Gamma|$.

To compute $\dim \mathcal{M}_{g,m}$, suppose for simplicity that $g \geqslant 2$. Then we find that

$$T_{[\Sigma,\vec{z}]}\mathcal{M}_{g,m} \cong H^1(T\Sigma) \oplus \bigoplus_{i=1}^{m} T_{z_i}\Sigma,$$

where the sheaf cohomology group $H^1(T\Sigma)$ parametrizes deformations of complex structure of Σ, and $T_{z_i}\Sigma$ parametrizes variations of the marked point z_i. Thus $\dim_{\mathbb{C}} \mathcal{M}_{g,m} = h^1(T\Sigma) + m$. But $H^0(T\Sigma) = 0$ as $g \geqslant 2$ and $H^k(T\Sigma) = 0$ for $k \geqslant 2$ as $\dim \Sigma = 1$, so $\dim H^1(T\Sigma) = -\chi(T\Sigma)$, and $\chi(T\Sigma) = 3 - 3g$ by the Riemann–Roch Theorem.

10.3. Examples

- $\mathcal{M}_{0,m} = \emptyset$ for $m = 0, 1, 2$ since $\mathrm{Aut}(\mathbb{CP}_1, \vec{z})$ is infinite, e.g. it is $\mathrm{PSL}(2, \mathbb{C})$ for $m = 0$.
- $\mathcal{M}_{0,3}$ is a single point, since any genus 0 curve with 3 marked points is isomorphic to $\bigl(\mathbb{CP}^1, ([1,0],[1,1],[0,1])\bigr)$.
- Suppose $[\Sigma, \vec{z}] \in \mathcal{M}_{0,4}$. Then there is a unique isomorphism $f : \Sigma \to \mathbb{CP}^1$ taking z_1, z_2, z_3 to $[1,0],[1,1],[0,1]$ respectively. Set $f(z_4) = [1, \lambda]$, for $\lambda \in \mathbb{C} \setminus \{0,1\}$. This gives an isomorphism $\mathcal{M}_{0,4} \cong \mathbb{C} \setminus \{0,1\}$. So $\mathcal{M}_{0,4}$ is *noncompact*, the complement of 3 points in $\mathbb{CP}^1$.

- Suppose $[\Sigma, \vec{z}] \in \mathcal{M}_{1,1}$. Choose a basis α, β for $H_1(\Sigma;\mathbb{Z})$. Then there is a unique $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and an isomorphism $f : \Sigma \to \mathbb{C}/\langle 1, \lambda\rangle_{\mathbb{Z}}$ with $f(z_1) = 0$, such that f identifies α with the loop $[0, 1]$ and β with the loop $\lambda[0, 1]$ in $\mathbb{C}/\langle 1, \lambda\rangle_{\mathbb{Z}}$. Choices of bases α, β for $H_1(\Sigma;\mathbb{Z})$ are parametrized by $GL(2;\mathbb{Z})$. So $\mathcal{M}_{1,1} \cong (\mathbb{C} \setminus \mathbb{R})/GL(2;\mathbb{Z})$. This is a noncompact complex 1-orbifold with two special orbifold points, one with group $\mathbb{Z}_4$ from $\lambda = i$, and one with group $\mathbb{Z}_6$ from $\lambda = e^{2\pi i/6}$. Every other point actually has stabilizer group $\mathbb{Z}_2$.

14.2. Compactification and Deligne–Mumford stable curves

To do Gromov–Witten theory, we need *compact* moduli spaces. So we need a *compactification* $\bar{\mathcal{M}}_{g,m}$ of $\mathcal{M}_{g,m}$. This must satisfy:

- $\bar{\mathcal{M}}_{g,m}$ is a compact, Hausdorff topological space containing $\mathcal{M}_{g,m}$ as an open subset.
- points of $\bar{\mathcal{M}}_{g,m} \setminus \mathcal{M}_{g,m}$ should have be interpreted as *singular* Riemann surfaces with marked points.
- $\bar{\mathcal{M}}_{g,m}$ is a complex orbifold.

In general, when compactifying moduli spaces, the compactification should be as close to being a smooth, oriented manifold as we can manage. In this case, we can make it a complex orbifold.
In algebraic geometry there are often several different ways of compactifying moduli spaces. In this case there is a clear best way to do it, using *Deligne–Mumford stable curves*.

A *prestable Riemann surface* Σ is a compact connected complex variety of dimension 1 whose only singular points are finitely many *nodes*, modelled on (0,0) in $\{(x,y) \in \mathbb{C}^2 : xy = 0\}$. Each such singular Σ is the limit as $t \to 0$ of a family of nonsingular Riemann surfaces Σ_t for $0 < |t| < \epsilon$ modelled on $\{(x,y) \in \mathbb{C}^2 : xy = t\}$ near each node of Σ.
We call Σ_t a *smoothing* of Σ. The *genus* of Σ is the genus of its smoothings Σ_t.
A *prestable Riemann surface* $(\Sigma, \vec{z})$ *with marked points* is a prestable Σ with $\vec{z} = (z_1, \ldots, z_m)$, where $z_1, \ldots, z_m$ are distinct smooth points (not nodes) of Σ. Define isomorphisms and $\mathrm{Aut}(\Sigma, \vec{z})$ as in the nonsingular case. We call $(\Sigma, \vec{z})$ *stable* if $\mathrm{Aut}(\Sigma, \vec{z})$ is finite.
The D–M moduli space $\overline{\mathcal{M}}_{g,m}$ is the set of isomorphism classes $[\Sigma, \vec{z}]$ of stable pairs $(\Sigma, \vec{z})$, where Σ is a prestable Riemann surface of genus g, and $\vec{z} = (z_1, \ldots, z_m)$ are distinct smooth points of Σ.

Theorem

$\overline{\mathcal{M}}_{g,m}$ is a compact, Hausdorff complex orbifold of complex dimension $3g + m - 3$.

The moduli spaces $\overline{\mathcal{M}}_{g,m}$ are very well-behaved, because of exactly the right choice of definition of singular curve. With (nearly) any other notion of singular curve, we would have lost compactness, or Hausdorffness, or smoothness.
The $\overline{\mathcal{M}}_{g,m}$ have been intensively studied, lots is known about their cohomology, etc.
Note that as $\overline{\mathcal{M}}_{g,m}$ is complex, it is *oriented* as a real orbifold.

Example

$\bar{\mathcal{M}}_{0,4}$ is $\mathbb{CP}^1$, with $\bar{\mathcal{M}}_{0,4} \setminus \mathcal{M}_{0,4}$ three points. These correspond to two $\mathbb{CP}^1$'s joined by a node, with two marked points in each $\mathbb{CP}^1$.

14.3. Moduli spaces of stable maps

Now let (S,ω) be a compact symplectic manifold, and J an almost complex structure compatible with ω. Fix $g,m\geqslant 0$ and $\beta\in H_2(S;\mathbb{Z})$. Consider triples $(\Sigma,\vec z,u)$ where $(\Sigma,\vec z)$ is a prestable Riemann surface of genus g (possibly singular) with marked points, and $u:\Sigma\to S$ a J-holomorphic map, with $u_*([\Sigma])=\beta$ in $H_2(S;\mathbb{Z})$.
An *isomorphism* between $(\Sigma,\vec z,u)$ and $(\Sigma',\vec z',u')$ is a biholomorphism $f:\Sigma\to\Sigma'$ with $f(z_i)=z_i'$ for $i=1,\ldots,m$ and $u'\circ f\equiv u$.
The *automorphism group* $\mathrm{Aut}(\Sigma,\vec z,u)$ is the set of isomorphisms from $(\Sigma,\vec z,u)$ to itself. We call $(\Sigma,\vec z,u)$ *stable* if $\mathrm{Aut}(\Sigma,\vec z,u)$ is finite. The *moduli space* $\bar{\mathcal{M}}_{g,m}(S,J,\beta)$ is the set of isomorphism classes $[\Sigma,\vec z,u]$ of stable triples $(\Sigma,\vec z,u)$, for Σ of genus g with m marked points $\vec z$, and $u_*([\Sigma])=\beta$ in $H_2(S;\mathbb{Z})$.
We also write $\mathcal{M}_{g,m}(S,J,\beta)$ for the subset of $[\Sigma,\vec z,u]$ with Σ nonsingular.

For $i = 1, \ldots, m$ define *evaluation maps* $\mathrm{ev}_i : \bar{\mathcal{M}}_{g,m}(S, J, \beta) \to S$ by $\mathrm{ev}_i : [\Sigma, \vec{z}, u] \mapsto u(z_i)$.
Define $\pi : \bar{\mathcal{M}}_{g,m}(S, J, \beta) \to \bar{\mathcal{M}}_{g,m}$ for $2g + m \geqslant 3$ by $\pi : [\Sigma, \vec{z}, u] \mapsto [\Sigma, \vec{z}]$, provided $(\Sigma, \vec{z})$ is stable. (If $(\Sigma, \vec{z})$ is unstable, map to the *stabilization* of $(\Sigma, \vec{z})$.)
There is a natural topology on $\bar{\mathcal{M}}_{g,m}(S, J, \beta)$ due to Gromov, called the C^∞ *topology*. It is derived from the notion of smooth family of prestable $(\Sigma, \vec{z})$ used to define the topology on $\bar{\mathcal{M}}_{g,m}$, and the C^∞ topology on smooth maps $u : \Sigma \to S$.

Theorem 14.1

$\bar{\mathcal{M}}_{g,m}(S, J, \beta)$ is a compact, Hausdorff topological space. Also $\mathrm{ev}_1, \ldots, \mathrm{ev}_m, \pi$ are continuous.

Both compactness and Hausdorffness in Theorem 14.1 are nontrivial. Hausdorffness really follows from the Hausdorffness of $\bar{\mathcal{M}}_{g,m}$. Compactness follows from the compactness of S, the compactness of $\bar{\mathcal{M}}_{g,m}$, the fixed homology class β, and the fact that J is compatible with a symplectic form ω, which bounds areas of curves.

Theorem 14.2 (Fukaya–Ono 1999; Hofer–Wysocki–Zehnder 2011)

We can make $\bar{\mathcal{M}}_{g,m}(S, J, \beta)$ into a compact, oriented Kuranishi space $\bar{\boldsymbol{\mathcal{M}}}_{g,m}(S, J, \beta)$, without boundary, of virtual dimension

$$2d = 2\bigl(c_1(S) \cdot \beta + (n-3)(1-g) + m\bigr), \tag{14.1}$$

where $\dim S = 2n$. Also $\mathrm{ev}_1, \ldots, \mathrm{ev}_m, \pi$ become 1-morphisms $\mathbf{ev}_i : \bar{\boldsymbol{\mathcal{M}}}_{g,m}(S, J, \beta) \to S$ and $\boldsymbol{\pi} : \bar{\boldsymbol{\mathcal{M}}}_{g,m}(S, J, \beta) \to \bar{\mathcal{M}}_{g,m}$.

For J-holomorphic maps $u : \Sigma \to S$ from a fixed Riemann surface Σ, or even from a varying, nonsingular Riemann surface Σ, this is fairly straightforward, given the technology we already discussed. Including singular curves is more difficult.

14.4. Virtual classes and Gromov–Witten invariants

We have now defined a compact, oriented Kuranishi space $\bar{\boldsymbol{\mathcal{M}}}_{g,m}(S,J,\beta)$ of dimension $2d$ in (14.1), and a 1-morphism

$$\mathbf{ev}_1 \times \cdots \times \mathbf{ev}_m \times \boldsymbol{\pi} : \bar{\boldsymbol{\mathcal{M}}}_{g,m}(S,J,\beta) \longrightarrow S^m \times \bar{\mathcal{M}}_{g,m}$$

if $2g+m \geqslant 3$, where $S^m \times \bar{\mathcal{M}}_{g,m}$ is an orbifold, or

$$\mathbf{ev}_1 \times \cdots \times \mathbf{ev}_m : \bar{\boldsymbol{\mathcal{M}}}_{g,m}(S,J,\beta) \to S^m$$

if $2g+m<3$, where S^m is a manifold.
As in §12.2, we can define a virtual class $[\bar{\boldsymbol{\mathcal{M}}}_{g,m}(S,J,\beta)]_{\rm virt}$ in $H_{2d}\big(S^m \times \bar{\mathcal{M}}_{g,m};\mathbb{Q}\big)$ or $H_{2d}\big(S^m;\mathbb{Q}\big)$.

Theorem 14.3 (Fukaya–Ono 1999)

These virtual classes $[\bar{\boldsymbol{\mathcal{M}}}_{g,m}(S,J,\beta)]_{\rm virt}$ are independent of the choice of almost complex structure J compatible with ω. They are also unchanged by continuous deformations of ω.

Sketch proof.

Let J_0, J_1 be possible almost complex structures. Choose a smooth family $J_t : t \in [0,1]$ of compatible almost complex structures joining them. Write $\bar{\mathcal{M}}_{g,m}(S,J_t : t\in[0,1],\beta)$ for the union of $\bar{\mathcal{M}}_{g,m}(S,J_t,\beta)$ over $t\in[0,1]$. This becomes a compact oriented Kuranishi space with boundary of virtual dimension $2d+1$, whose boundary is $\bar{\boldsymbol{\mathcal{M}}}_{g,m}(S,J_1,\beta) \amalg -\bar{\boldsymbol{\mathcal{M}}}_{g,m}(S,J_0,\beta)$.
Construct a virtual chain for $\bar{\boldsymbol{\mathcal{M}}}_{g,m}(S,J_t : t\in[0,1],\beta)$. This is a $(2d+1)$-chain on $S^m \times \bar{\mathcal{M}}_{g,m}$ whose boundary is the difference of virtual cycles for $\bar{\boldsymbol{\mathcal{M}}}_{g,m}(S,J_1,\beta)$ and $\bar{\boldsymbol{\mathcal{M}}}_{g,m}(S,J_0,\beta)$. Thus these cycles are homologous, and their homology classes, the virtual classes $[\bar{\boldsymbol{\mathcal{M}}}_{g,m}(S,J_i,\beta)]_{\rm virt}$ are the same. The same argument works for continuous deformations of ω. □

Gromov–Witten invariants

Gromov–Witten invariants are basically the virtual classes $[\bar{\mathcal{M}}_{g,m}(S,J,\beta)]_{\rm virt}$. But it is conventional to define them as *maps on cohomology*, rather than as homology classes. We follow Cox and Katz §7. Since $\bar{\mathcal{M}}_{g,m}$ is a compact oriented orbifold of real dimension $6g+2m-6$, Poincaré duality gives an isomorphism

$$H_l(\bar{\mathcal{M}}_{g,m};\mathbb{Q}) \cong H^{6g+2m-6-l}(\bar{\mathcal{M}}_{g,m};\mathbb{Q}). \tag{14.2}$$

For $g,m\geqslant 0$ and $\beta\in H_2(S;\mathbb{Z})$, the *Gromov–Witten invariant*

$$\langle I_{g,m,\beta}\rangle : H^*(S;\mathbb{Q})^{\otimes^m}\rightarrow\mathbb{Q}$$

is the linear map corresponding to the virtual cycle $[\bar{\mathcal{M}}_{g,m}(S,J,\beta)]_{\rm virt}$ in $H_{2d}\bigl(S^m;\mathbb{Q}\bigr)$ under the Künneth isomorphism

$$H_*\bigl(S^m;\mathbb{Q}\bigr)\cong\bigl(H^*(S;\mathbb{Q})^{\otimes^m}\bigr)^*$$

This is zero on $H^{k_1}(S;\mathbb{Q})\otimes\cdots\otimes H^{k_m}(S;\mathbb{Q})$ unless $k_1+\cdots+k_m=2d$.

Gromov–Witten classes

For $2g+m\geqslant 3$ and $\beta\in H_2(S;\mathbb{Z})$, the *Gromov–Witten class*

$$I_{g,m,\beta} : H^*(S;\mathbb{Q})^{\otimes^m}\rightarrow H^*(\bar{\mathcal{M}}_{g,m};\mathbb{Q})$$

is the linear map corresponding to $[\bar{\mathcal{M}}_{g,m}(S,J,\beta)]_{\rm virt}$ under

$$H_*\bigl(S^m\times\bar{\mathcal{M}}_{g,m};\mathbb{Q}\bigr)\cong\bigl(H^*(S;\mathbb{Q})^{\otimes^m}\bigr)^*\otimes H^{6g+2m-6-*}(\bar{\mathcal{M}}_{g,m};\mathbb{Q}),$$

using Künneth again and (14.2). The relation between G–W invariants $\langle I_{g,m,\beta}\rangle$ and G–W classes $I_{g,m,\beta}$ is

$$\langle I_{g,m,\beta}\rangle=\int_{\bar{\mathcal{M}}_{g,m}} I_{g,m,\beta},$$

that is, $\langle I_{g,m,\beta}\rangle$ is the contraction of $I_{g,m,\beta}$ with the fundamental class $[\bar{\mathcal{M}}_{g,m}]$. Gromov–Witten classes satisfy a system of axioms (Kontsevich and Manin 1994). Using them we can define *quantum cohomology* of symplectic manifolds.

The Splitting Axiom

Suppose $[\Sigma^1,\vec z^1]\in\overline{\mathcal M}_{g^1,m^1+1}$ and $[\Sigma^2,\vec z^2]\in\overline{\mathcal M}_{g^2,m^2+1}$. Then we can glue Σ_1,Σ_2 together at marked points $z^1_{m^1+1},z^2_{m^2+1}$ to get Σ with a node at $z^1_{m^1+1}=z^2_{m^2+1}$.

The Splitting Axiom

This Σ has genus $g=g^1+g^2$ and $m=m^1+m^2$ remaining marked points $z^1_1,\ldots,z^1_{m^1}$ from $(\Sigma^1,\vec z^1)$ and $z^2_1,\ldots,z^2_{m^2}$ from $(\Sigma^2,\vec z^2)$. Define $\vec z=(z^1_1,\ldots,z^1_{m^1},z^2_1,\ldots,z^2_{m^2})$. Then $[\Sigma,\vec z]\in\overline{\mathcal M}_{g,m}$. This defines a map

$$\varphi:\overline{\mathcal M}_{g^1,m^1+1}\times\overline{\mathcal M}_{g^2,m^2+1}\to\overline{\mathcal M}_{g,m}.$$

Choose a basis $(T_i)_{i=1}^N$ for $H^*(S;\mathbb Q)$, and let $(T^j)_{j=1}^N$ be the dual basis under the cup product, that is, $T_i\cup T^j=\delta_i^j$.
Then the Splitting Axiom says that

$$\begin{aligned}\varphi^*\bigl(I_{g,m,\beta}(\alpha^1_1,\ldots,\alpha^1_{m^1},\alpha^2_1,\ldots,\alpha^2_{m^2})\bigr)=\\ \sum_{\beta=\beta^1+\beta^2}\textstyle\sum_{i=1}^N I_{g^1,m^1,\beta^1}(\alpha^1_1,\ldots,\alpha^1_{m^1},T_i)\otimes\\ I_{g^2,m^2,\beta^2}(\alpha^2_1,\ldots,\alpha^2_{m^2},T^i).\end{aligned}$$

The Splitting Axiom

Here is how to understand this. Let Δ_S be the diagonal $\{(p,p) : p \in S\}$ in $S \times S$. Then $\sum_{i=1}^N T_i \otimes T^i$ in $H^*(S;\mathbb{Q}) \otimes H^*(S;\mathbb{Q})$ is Poincaré dual to $[\Delta_S]$ in $H_*(S \times S;\mathbb{Q}) \cong H_*(S;\mathbb{Q}) \otimes H_*(S;\mathbb{Q})$.
Thus the term

$$\sum_{i=1}^N I_{g^1,m^1,\beta^1}(\alpha^1_1,\ldots,\alpha^1_{m^1},T_i) \otimes I_{g^2,m^2,\beta^2}(\alpha^2_1,\ldots,\alpha^2_{m^2},T^i)$$

'counts' pairs of curves $u^1 : \Sigma^1 \to S$ and $u^2 : \Sigma^2 \to S$, with genera g^1, g^2 and homology classes β^1,β^2, such that $u^a(\Sigma^a)$ intersects cycles $C^a_1,\ldots,C^a_{m^a}$ Poincaré dual to $\alpha^a_1,\ldots,\alpha^a_{m^a}$, and also $u^1(\Sigma^1) \times u^2(\Sigma^2)$ intersects the diagonal Δ_S in $S \times S$. This last condition means that $u^1(\Sigma^1)$ and $u^2(\Sigma^2)$ intersect in S. But then we can glue Σ^1, Σ^2 at their intersection point to get a nodal curve Σ, genus $g = g^1 + g^2$, class $\beta = \beta^1 + \beta^2$.