POISSON-RIEMANNIAN GEOMETRY

Shahn Majid (QMUL) GQTA Sandbjerg 2018

⓪ Quantum Riemannian geometry on any algebra $(A, \Omega, \mathrm{d}, g, \nabla, ...)$

LTCC lectures 2011 (a bimodule approach — DVM) & Book w/ Beggs 2019

① Quantum geometry of quantum groups

Alg. Repn. Theory, 20 (2017)

② Semiclassicalisation of quantum Riemannian geometry

$(C^\infty(M), \omega, g, \nabla)$ ω Poisson tensor g metric ∇ (flat) Poisson conn

w/ Beggs, J. Geom. Phys. 114 (2017)

If does not exist flat conn => $\left\{ \begin{array}{l} \text{non assoc ext. algebra} \\ \text{extra cotangent dimensions} \end{array} \right.$

E.g. $M = \mathrm{ch}(S) = \{\pi_1(S) \to G\}/G = \{G - \mathrm{bun},\ \text{flat conn}\}$

ω Poisson structure (ABGN) g Kahler

Quantum differentials on an algebra A

Classically, $C^\infty(M)=\Omega^0(M)\subset\Omega(M)=\oplus_i\Omega^i(M)$

Ω^1 space of 1-forms, e.g. `differentials' $\mathrm{d}f=\sum_i \frac{\partial f}{\partial x^i}\mathrm{d}x^i$

$f\mathrm{d}g=(\mathrm{d}g)f\in\Omega^1$

$$\wedge:\Omega\otimes_A\Omega\to\Omega,\quad \mathrm{d}(\omega\wedge\eta)=(\mathrm{d}\omega)\wedge\eta+(-1)^{|\omega|}\omega\wedge\mathrm{d}\eta$$

$\omega\wedge\eta=(-1)^{|\omega||\eta|}\eta\wedge\omega,\quad \mathrm{d}^2=0$ `graded Leibniz rule'

- algebra *A* over k we drop the (graded) commutativity, just keep:

Ω^1	a((db)c)=(a(db))c	`bimodule'
$\mathrm{d}:A\to\Omega^1$	d(ab)=(da)b+a(db)	`Leibniz rule'
$\{\sum a\mathrm{d}b\}=\Omega^1$		`surjectivity'
$\ker\mathrm{d}=k.1$		(`connected')

- require this to extend to a DGA $\Omega=T_A\Omega^1/\mathcal{I}=\oplus_n\Omega^n,\quad \mathrm{d}^2=0$
- inner if exists $\theta\in\Omega^1,\quad \mathrm{d}=[\theta,\ \}$

Nice problem: take your favourite algebra and classify all differential structures (perhaps with some symmetry)

Thm

bicovariant surjective

$$\Omega^1(U(\mathfrak{g})) \longleftrightarrow \zeta \in Z^1(\mathfrak{g}, \Lambda^1)$$

$$\mathrm{d}x = 1 \otimes \zeta(x), \ \Omega^1 = U(\mathfrak{g}) \otimes \Lambda^1$$

e.g. bicovariant connected classical dim $\longleftrightarrow$ pre-Lie algebra

$$\circ : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g} \qquad [x, y] = x \circ y - y \circ x$$

$$(x \circ y) \circ z - (y \circ x) \circ z = x \circ (y \circ z) - y \circ (x \circ z)$$

$$\Lambda^1 = \{\mathrm{d}y \mid y \in \mathfrak{g}\} \cong \mathfrak{g} \quad [x, \mathrm{d}y] = \mathrm{d}(x \circ y)$$

$\Rightarrow \ \Omega(U(\mathfrak{g}))$ by skew-symmetrisation of products of Λ^1

- Example $\mathfrak{g} : \ [r, t] = \lambda r$ (see later)
- Example $\mathfrak{g} = \mathrm{Vect}(M)$ and torsion free flat connection

$$x \circ y = \nabla_x y, \qquad \nabla_{[x,y]} z = \nabla_x \nabla_y z - \nabla_y \nabla_x z$$

$$A = U(\mathrm{diff}(M)) \qquad [x, y] = \nabla_x y - \nabla_y x$$

- Example $\mathfrak{g} = V, \quad [\ ,\ ] = 0, \quad (V, \circ)$ comm associative algebra

e.g. $V = C^\infty(M) \qquad A = C_{\rm poly}(V) \qquad [f, \mathrm{d}g] = \mathrm{d}(fg)$

e.g. $V = \mathbb{C}.x, \quad x \circ x = \lambda x, \quad A = \mathbb{C}[x], \quad [x, \mathrm{d}x] = \lambda \mathrm{d}x$

=> $$\mathrm{d}f(x) = \frac{f(x) - f(x - \lambda x)}{\lambda}\mathrm{d}x$$

$$\mathrm{d}x^2 = (\mathrm{d}x)x + x\mathrm{d}x = 2x\mathrm{d}x - \lambda\mathrm{d}x = (2x - \lambda)\mathrm{d}x$$

Propn X discrete set $\Omega^1(C(X)) \quad \longleftrightarrow$ directed graphs on X

$$\Omega^1 = \mathrm{span}_k\{\omega_{x\to y}\}$$

$$f.\omega_{x\to y} = f(x)\omega_{x\to y}, \omega_{x\to y}.f = f(y)\omega_{x\to y} \qquad \mathrm{d}f = \sum_{x\to y}(f(y) - f(x))\omega_{x\to y}$$

If a graph is bidirected, define

$$g = \sum_{x\to y} g_{x\to y}\omega_{x\to y} \otimes_{C(X)} \omega_{y\to x} \qquad g_{x\to y} \in k$$ `metric lengths'

Quantum metrics

$$g = g_{\mu\nu} \mathrm{d}x^\mu \otimes_A \mathrm{d}x^\nu$$

$$g \in \Omega^1 \underset{A}{\otimes} \Omega^1 \qquad \wedge(g) = 0$$

`quantum symmetric'

invertible in the sense exists inverse: $(\ , \ ) : \Omega^1 \underset{A}{\otimes} \Omega^1 \to A$

$$((\ ,\ ) \otimes \mathrm{id})(\omega \otimes g) = \omega = (\mathrm{id} \otimes (\ ,\ ))(g \otimes \omega), \quad \forall \omega \in \Omega^1$$

$$a(\omega, \eta) = (a\omega, \eta), \quad (\omega, \eta)a = (\omega, \eta a)$$

`bimodule map (tensorial)'

need this to be able to contract/ `raise/lower' via metric, eg to have well defined contraction:

$$(\ ,\ ) \otimes \mathrm{id} : \quad \Omega^1 \underset{A}{\otimes} \Omega^1 \underset{A}{\otimes} \Omega^1 \to \Omega^1$$

" $T_{\mu\nu\rho} \mapsto g^{\mu\nu} T_{\mu\nu\rho}$ "

but

$$(\omega, g^1)g^2 = \omega \qquad g = g^1 \underset{A}{\otimes} g^2$$

$$\implies (\omega, g^1)g^2 a = \omega a = (\omega a, g^1)g^2 = (\omega, a g^1)g^2$$

$$\implies \qquad ag = ga, \quad \forall a \in A$$

need metric to be central

Connections and curvature

Classically, a connection assigns a covariant derivative

$$\nabla_x : \mathrm{Vect}(M) \to \mathrm{Vect}(M), \quad \forall x \in \mathrm{Vect}(M)$$

(Christoffel symbols)

$$\nabla_x : \Omega^1(M) \to \Omega^1(M)$$

$$\nabla \mathrm{d}x^\mu = -\Gamma^\mu{}_{\nu\rho} \mathrm{d}x^\nu \otimes_A \mathrm{d}x^\rho$$

Similarly for any differential algebra $(A, \Omega^1, \mathrm{d})$

bimodule connection: $\nabla : \Omega^1 \to \Omega^1 \underset{A}{\otimes} \Omega^1 \qquad \sigma : \Omega^1 \underset{A}{\otimes} \Omega^1 \to \Omega^1 \underset{A}{\otimes} \Omega^1$

$$\nabla(f\omega) = \mathrm{d}f \otimes \omega + f\nabla\omega \qquad \nabla(\omega f) = \sigma(\omega \otimes \mathrm{d}f) + (\nabla\omega)f$$

(Quillen, Karoubi,...) (Michor, Dubois-Violette, ...)

such connections extend to tensor products

$$\omega \otimes \eta \in \Omega^1 \underset{A}{\otimes} \Omega^1 \qquad \nabla(\omega \otimes \eta) = \nabla\omega \otimes \eta + (\sigma \otimes \mathrm{id})(\omega \otimes \nabla\eta)$$

more generally $\nabla_E : E \to \Omega^1 \otimes_A E, \quad \sigma_E : E \otimes_A \Omega^1 \to \Omega^1 \otimes_A E$

${}_A\mathcal{E}_A = \{(E, \nabla_E, \sigma_E)\}$ is a monoidal category by $\otimes_A$

`metric compatible' now makes sense $\nabla g=0$

torsion free also makes sense $T_\nabla:\Omega^1\to\Omega^2 \qquad T_\nabla=\wedge\nabla-\mathrm{d}$

- quantum Levi-Civita connection (QLC) $T_\nabla=\nabla g=0$

- `weak quantum Levi-Civita' needs only a left connection, torsion free and cotorsion free: $\mathrm{co}T_\nabla=(\mathrm{d}\otimes\mathrm{id}-(\wedge\otimes_A\mathrm{id})(\mathrm{id}\otimes_A\nabla))g=0$

- Curvature

$$R_\nabla:\Omega^1\to\Omega^2\mathop{\otimes}_A\Omega^1 \qquad R_\nabla=(\mathrm{d}\mathop{\otimes}_A\mathrm{id}-(\wedge\mathop{\otimes}_A\mathrm{id})(\mathrm{id}\mathop{\otimes}_A\nabla))\nabla$$

Lemma: (1st Bianchi identity) $\wedge(R_\nabla)=\mathrm{d}\circ T_\nabla-(\wedge\otimes\mathrm{id})(\mathrm{id}\otimes T_\nabla)\nabla$

- Laplacian $\Delta:A\to A, \quad \Delta=(\ ,\ )\nabla\mathrm{d}$

2.1 Example of 2D nonabelian Lie algebra

$$\mathfrak{g} : [r,t] = r \qquad A: \quad [r,t] = \lambda r$$

$i)\quad t\circ r = -r, \quad t\circ t = \alpha t$

$ii)\quad r\circ t = \beta r, \quad t\circ r = (\beta-1)r, \quad t\circ t = \beta t$

} just two essentially different differential structures

$iii)\quad t\circ r = -r, \quad t\circ t = r - t$

$iv)\quad r\circ r = t, \quad t\circ r = -r, \quad t\circ t = -2t$

$v)\quad r\circ t = r, \quad t\circ t = r + t$ (Burde)

} included in i), ii) if we allow logarithms etc

Phys. Rev. D 91 (2015)

Case (i) => unique form of metric AdS or dS, unique QLC with classical limit

Case (ii) $\beta = 1$ => unique form of quantum metric

$$g = \mathrm{d}r\otimes\mathrm{d}r + b(v^*\otimes v + \lambda(\mathrm{d}r\otimes v - v^*\otimes\mathrm{d}r))$$

$$v = r\mathrm{d}t - t\mathrm{d}r, \quad v^* = (\mathrm{d}t)r - t\mathrm{d}r, \quad b\in\mathbb{R} \quad b\neq 0$$

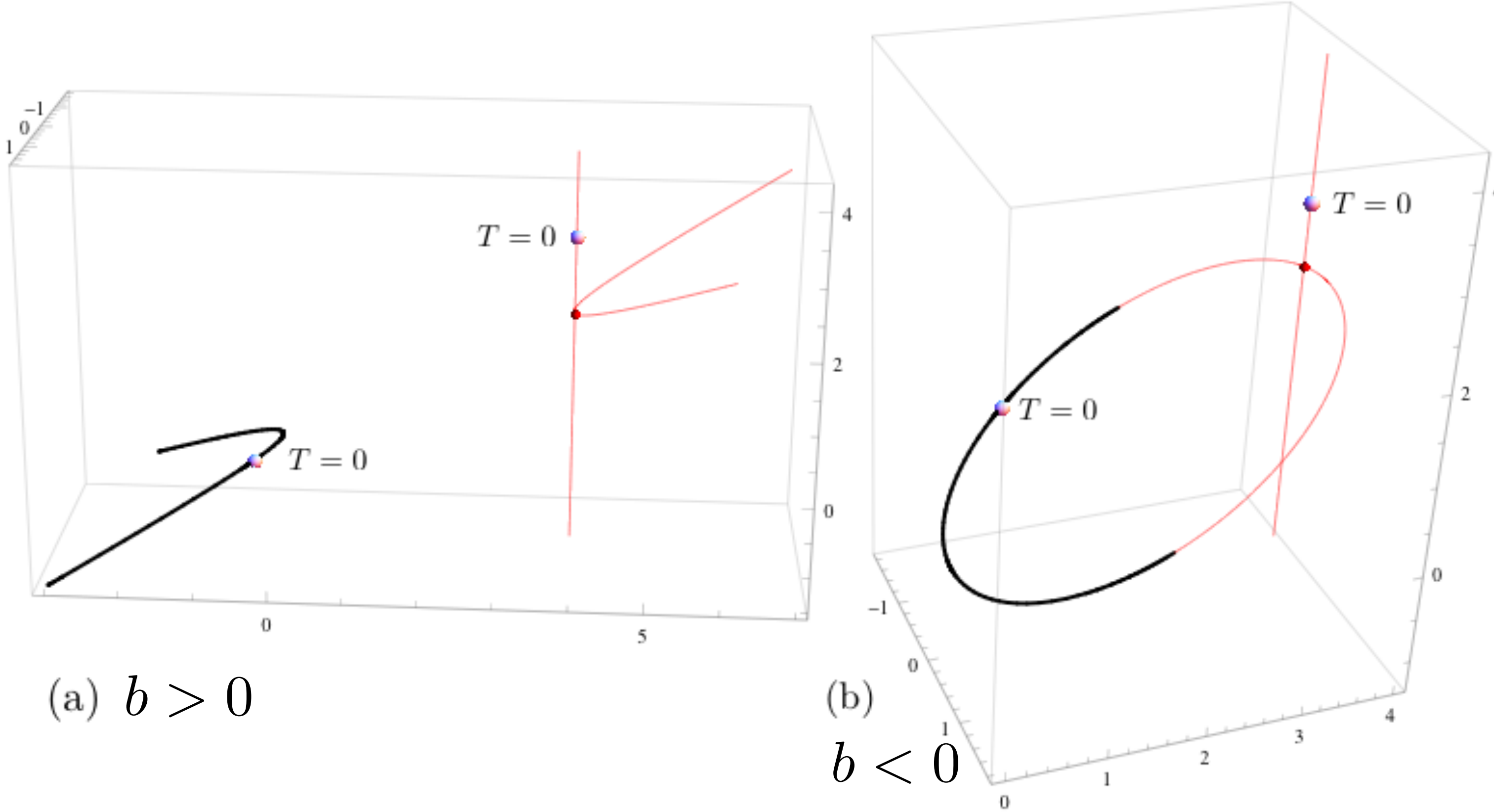
in classical limit $\mathrm{Ricci} = \frac{g}{r^2}$ all timelike geodesic pulled back to r=0

=> Unique Levi-Civita soln with classical limit:

$$\nabla\mathrm{d}r = \frac{1}{r}\left(v - \frac{\lambda\mathrm{d}r}{2}\right)\otimes\left(\left(\frac{8b}{4+7b\lambda^2}\right)v - \left(\frac{12b\lambda}{4+7b\lambda^2}\right)\mathrm{d}r\right)$$

Class. Quant. Gravity 31 (2014) (w/ Beggs)

=> Moduli of real quantum metric-compatible ∇ a line + conic



- black parts have classical limit as $\lambda \to 0$
- red parts blow up as $\lambda \to 0$ so not visible classically
- in each case a unique `q. Levi-Civita point' where torsion T=0

$$\nabla \mathrm{d}r = \frac{bv}{r} \otimes \left((\frac{1}{1+b\lambda^2})v - (\frac{2}{\lambda})\mathrm{d}r \right) + (\frac{2+b\lambda^2}{r(1+b\lambda^2)})\mathrm{d}r \otimes \left(-(\frac{1}{\lambda})v + (\frac{3}{2})\mathrm{d}r \right)$$

2.2 Example of quantum geometry of a quadrilateral

Cayley graph on ad-stable set generators $\mathcal{C}$ of a group X

edges: $x \to xa, a \in \mathcal{C}$

$$e_a f = R_a(f) e_a, \quad \mathrm{d}f = \sum_{a\in\mathcal{C}} \partial^a(f) e_a$$

$$\partial^a = R_a - \mathrm{id}$$

left-invariant 1-forms: $e_a = \sum_{x\in X} \omega_{x\to xa}$

$$\Omega^1 \Rightarrow \Omega \qquad \mathrm{d} = [\theta, \ \} \quad \theta = \sum_{a\in\mathcal{C}} e_a$$

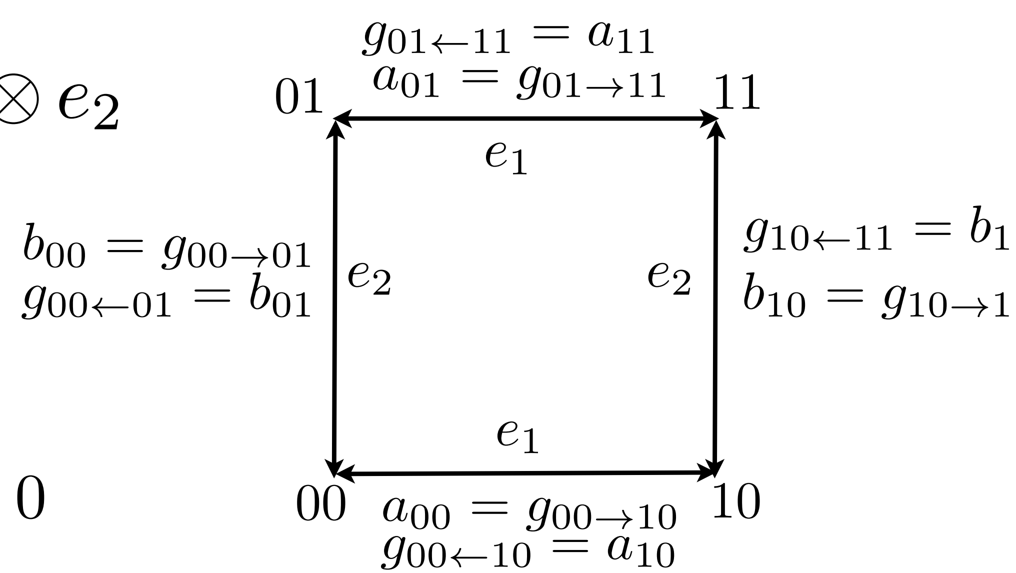
$$X = \mathbb{Z}_2 \times \mathbb{Z}_2 \qquad e_a^2 = 0, \ e_a e_b + e_b e_a = 0$$

$$\mathcal{C} = \{1 = (1,0), 2 = (0,1)\} \qquad \mathrm{d}e_a = 0$$

=> metric $g = a e_1 \otimes e_1 + b e_2 \otimes e_2$

for some functions a,b

It is natural to suppose g symmetric `lengths': $\partial^1 a = \partial^2 b = 0$



=> 1-parameter moduli space of torsion free metric compatible connection, with curvature:

$$\nabla\omega = \theta\otimes\omega - \sigma(\omega\otimes\theta)$$

$$Q = (q, q^{-1}, q^{-1}, q)$$

$$\sigma = \begin{pmatrix} -Q^{-1} & 0 & 0 & \frac{a(R_1\alpha-1)}{b} \\ 0 & \alpha-1 & \beta & 0 \\ 0 & \alpha & \beta-1 & 0 \\ \frac{b(R_2\beta-1)}{a} & 0 & 0 & Q \end{pmatrix}$$

cf `8-vertex R-matrix'

$$\alpha = (\frac{a_{01}}{a_{00}}, 1, 1, \frac{a_{00}}{a_{01}}) \qquad \beta = (1, \frac{b_{10}}{b_{00}}, \frac{b_{00}}{b_{10}}, 1)$$

=> connection

$$\nabla e_1 = (1+Q^{-1})e_1\otimes e_1 + (1-\alpha)(e_1\otimes e_2 + e_2\otimes e_1) - \frac{b}{a}(R_2\beta-1)e_2\otimes e_2.$$

$$\nabla e_2 = -\frac{a}{b}(R_1\alpha-1)e_1\otimes e_1 + (1-\beta)(e_1\otimes e_2 + e_2\otimes e_1) + (1-Q)e_2\otimes e_2.$$

curvature $$R_\nabla e_1 = \Big(Q^{-1}R_1\alpha - Q\alpha + (1-\alpha)(R_1\beta-1) + \frac{R_2 a}{a}(R_2\beta-1)(R_2R_1\alpha-1)\Big)\mathrm{Vol}\otimes e_1$$

$$+\Big(Q^{-1}(1-\alpha) + \alpha(R_2\alpha-1) + Q^{-1}\frac{R_1 b}{a}(\beta^{-1}-1)) + \frac{b}{a}(R_2\beta-1)R_2\beta\Big)\mathrm{Vol}\otimes e_2$$

=> geometric Laplacian

$$\Delta f = (\ ,\ )\nabla(\partial_i f e_i) = -\frac{2}{a}\partial_1 f - \frac{2}{b}\partial_2 f + \partial_i f(\ ,\ )\nabla e_i = \left(\frac{Q^{-1} - R_2\beta}{a}\right)\partial_1 f - \left(\frac{Q + R_1\alpha}{b}\right)\partial_2 f$$

and Ricci curvature, e.g.

$$\mathrm{Ricci}_{q=1} = \frac{1}{2}\begin{pmatrix} \frac{1}{b}(-\frac{\partial_2 a}{\alpha} + \chi\frac{\partial_1 b}{\beta}) & -\frac{\partial_1 b}{b}(\alpha + \frac{1}{\alpha} - \chi - 2) \\ -\frac{\partial_2 a}{a}(\beta + \frac{1}{\beta} - \chi - 2) & \frac{1}{a}(-\frac{\partial_2 a}{\alpha} + \chi\frac{\partial_1 b}{\beta}) \end{pmatrix} \qquad \chi = (1, -1, -1, 1)$$

$$S = -\frac{1}{4ab}\left((3 + q + (1 - q)\chi)\frac{\partial_2 a}{\alpha} + (1 - q^{-1} - (3 + q^{-1})\chi)\frac{\partial_1 b}{\beta}\right)$$

Choice of measure $\mu = |ab| = ab$ =>

$$\int S = \sum_{\mathbb{Z}_2\times\mathbb{Z}_2} \mu S = (a_{00} - a_{01})^2(\frac{1}{a_{00}} + \frac{1}{a_{01}}) + (b_{00} - b_{10})^2(\frac{1}{b_{00}} + \frac{1}{b_{10}})$$

measures the `energy' in the gravitational field. `Bathtub' shape minimised at a, b constant (`rectangular' geometry)

Note for bicovariant calculus on any Hopf algebra braiding Ψ
=> \`antisymmetrization' => canonical (Ω, d) from (Ω^1, d)

Nonabelian group example $G = S_3 = \langle u, v\rangle / u^2 = v^2 = e,\ uvu = vuv$

$$\mathcal{C} = 2 - \text{cycles}, \quad \Lambda^1 = \{e_u, e_v, e_w\} \quad w = uvu \quad \dim(\Lambda) = 1:3:4:3:1$$

$$e_u \wedge e_v + e_v \wedge e_w + e_w \wedge e_u = 0, \quad e_v \wedge e_u + e_u \wedge e_w + e_w \wedge e_v = 0, \quad e_u^2 = e_v^2 = e_w^2 = 0$$

$$\mathrm{d}e_u + e_v \wedge e_w + e_w \wedge e_v = 0, \quad \mathrm{d}e_v + e_w \wedge e_u + e_u \wedge e_w = 0, \quad \mathrm{d}e_w + e_u \wedge e_v + e_v \wedge e_u = 0$$

$$\mathrm{Vol} := e_u \wedge e_v \wedge e_u \wedge e_w = e_v \wedge e_u \wedge e_v \wedge e_w = -e_w \wedge e_u \wedge e_v \wedge e_u = -e_w \wedge e_v \wedge e_u \wedge e_v$$

$$g = \sum_a e_a \otimes e_a \quad \Rightarrow \quad \nabla e_u = (3+\lambda) e_u \otimes e_u + (1+\mu)\theta \otimes \theta - \Psi^{-1}(e_u \otimes \theta)$$

$$\theta = e_u + e_v + e_w$$

2-param WQLC but no QLC; 1-param Einstein

Remark:

G	S_2	S_3	S_4	S_5
Top deg	1	4	12	40

... same numbers as number of indecomposables of preprojective algebra/ components of Lusztig-Kashiwara canonical basis for type An...

2.3 quantum geometry of 2x2 matrices

Prop $\Omega^1(M_2(\mathbb{C}))$ are inner, parallelizable and up to isom:

$\Omega^1 = M_2 \oplus M_2 \oplus M_2$ unique universal calc

$\Omega^1 = M_2 \oplus M_2$ $\mathbb{CP}^2$ $s = 1 \oplus 0, t = 0 \oplus 1$ $\mathrm{d}a = [\theta_s, a]s + [\theta_t, a]t$

$\Omega^1 = M_2$ $\mathbb{CP}^2$

In any exterior algebra s, t eg $s^2 = t^2 = 0$ => $\dim_{M_2}(\Omega) = 1:2:1$

e.g. $\mathrm{d}a = [E_{12}, a]s + [E_{21}, a]t$

=> $\mathrm{H}^0_{\mathrm{dR}}(M_2(\mathbb{C})) = \mathbb{C}.1, \quad \mathrm{H}^1_{\mathrm{dR}}(M_2(\mathbb{C})) = \mathbb{C}E_{21}s \oplus \mathbb{C}E_{12}t, \quad \mathrm{H}^2_{\mathrm{dR}}(M_2(\mathbb{C})) = \mathbb{C}s \wedge t$

e.g. $g = s \otimes s + t \otimes t$ => incl. 3-param family QLCs containing

$$\nabla s = 2E_{21}t \otimes s, \quad \nabla t = 2E_{12}s \otimes t, \quad \sigma(s^i \otimes s^j) = (-1)^{i-j}s^j \otimes s^i$$

$$R_\nabla s = 2\sigma_3 s \wedge t \otimes s, \;\; R_\nabla t = -2\sigma_3 s \wedge t \otimes t$$

2.4 Quantum group $\mathbb{C}_q[SL_2]$

$$\dim(\Omega) = 1:4:6:4:1$$

$$\mathbb{C} < a,b,c,d > /ba = qab,\ dc = qcd,\ cb - bc,\ db = qbd,\ ca = qac,\ ad - q^{-1}bc = 1 = da - qbc$$

$$\Omega^1 = \mathbb{C}_q[SL_2].\Lambda^1$$

$$\Lambda^1 = \{e_b, e_c, e_z, \theta\} \qquad \{e_b, e_c\} = 0 \qquad e_b^2 = e_c^2 = \theta^2 = 0$$ (Woronowicz '89)

$$e_z \wedge e_c + q^2 e_c \wedge e_z = 0, \quad e_b \wedge e_z + q^2 e_z \wedge e_b = 0, \quad e_z \wedge e_z = (1 - q^{-4}) e_c \wedge e_b$$

$$\mathrm{d}\theta = 0, \quad \mathrm{d}e_c = q^2 e_c \wedge e_z, \quad \mathrm{d}e_b = q^2 e_z \wedge e_b, \quad \mathrm{d}e_z = (q^{-2} + 1) e_b \wedge e_c.$$

$$\mathrm{Vol} = e_b \wedge e_c \wedge e_z \wedge \theta$$

$$g = q^2 e_c \otimes e_b + e_c \otimes e_b + \frac{q^2}{(2)_q}(e_z \otimes e_z - \theta \otimes \theta)$$

unique invariant
`Killing' metric

WQLC w/ $\nabla\theta = 0$

$$\lambda = 1 - q^{-2}$$

$$\nabla e_a = -\nabla e_d = \frac{1}{(2)_{q^2}}\left(e_b \otimes e_c - e_c \otimes e_b - \lambda \frac{q^3}{(2)_q} e_z \otimes e_z\right) = \frac{q}{(2)_q} \nabla e_z$$

$$(n)_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

$$\nabla e_b = \frac{1}{(2)_{q^2}}(e_z \otimes e_b - q^2 e_b \otimes e_z)$$

=> q-Laplacian Δ eigenvalues

$$\nabla e_c = \frac{1}{(2)_{q^2}}(-q^2 e_z \otimes e_c + e_c \otimes e_z)$$

$$(j)_q (j+1)_q$$

3. Braided-Hopf Algebra Fourier Transform

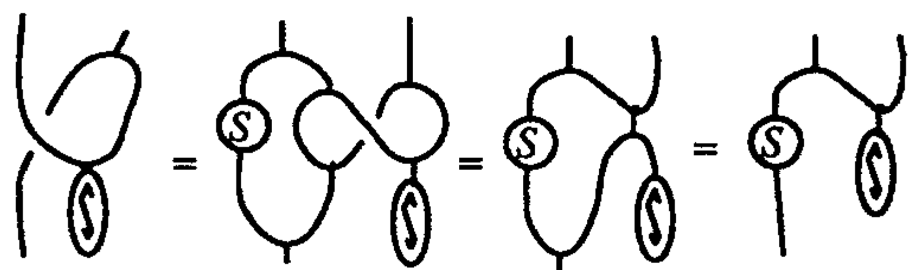
Suppose (1) B has a left integral $\int B \to \underline{1}$ $\qquad (\mathrm{id}\otimes\int)\Delta = \eta\otimes\int$

(2) B has a left dual

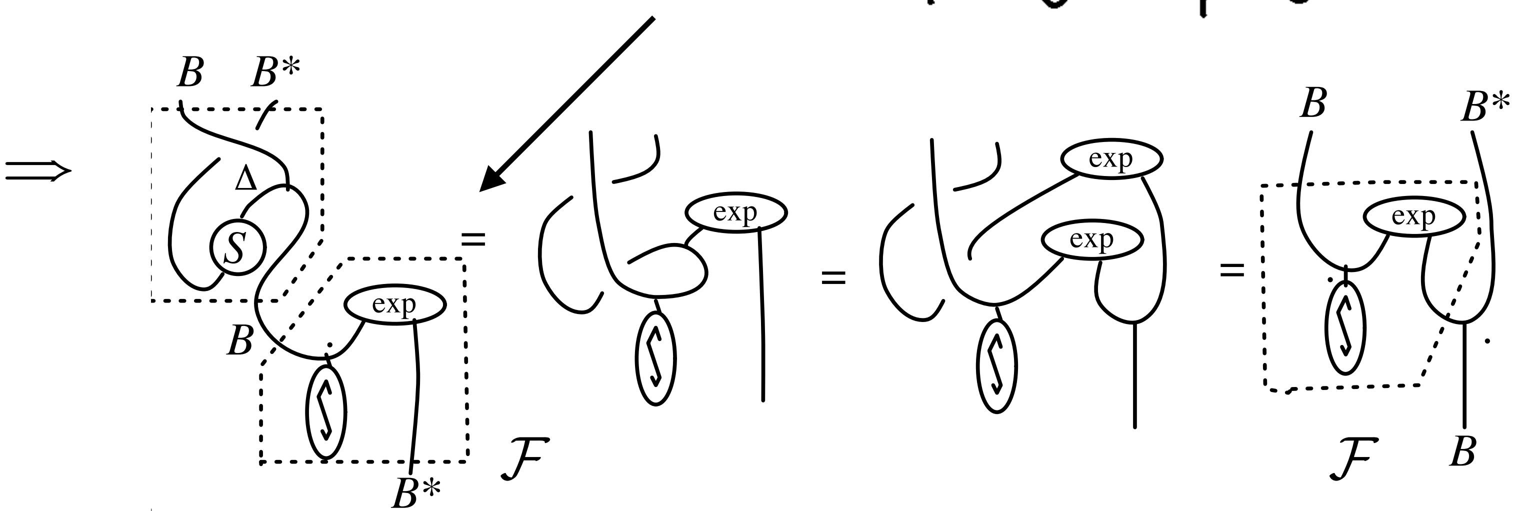
$$\mathrm{ev} = \cup : B^*\otimes B \to \underline{1}, \quad \mathrm{coev} = \cap = \exp : \underline{1} \to B\otimes B^*$$

$\Longrightarrow$ braided Fourier transform $\qquad \mathcal{F} : B \to B^*, \quad \mathcal{F}\circ \mathrm{Reg} = \cdot(\mathcal{F}\otimes\mathrm{id})$

Proof (Kempf & SM '94)



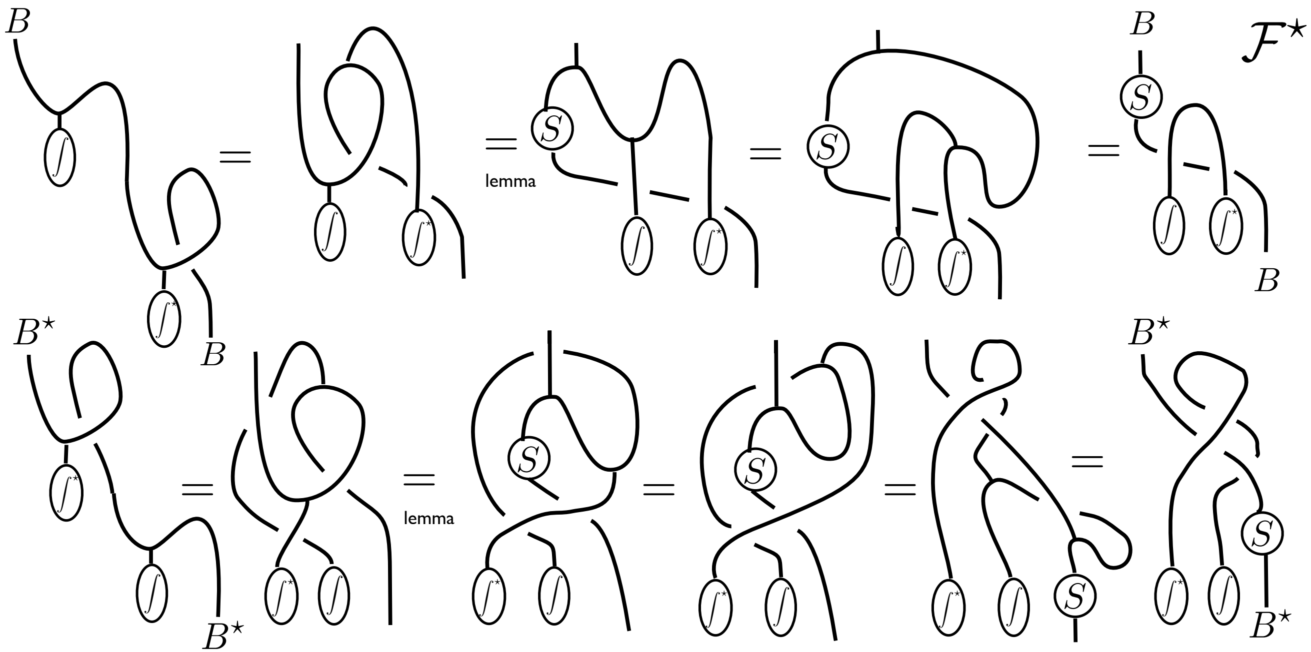
$\Longrightarrow$



$\mathcal{F}$ $\qquad\qquad$ $\mathcal{F}$

Suppose B* also has an integral, define $\mathcal{F}^*$ then (new)

$$\mathcal{F}^\star\mathcal{F} = \mu S, \quad \mu := \left(\int \otimes \int^*\right)\exp$$



If the integrals are both unimodular and morphisms then $\mathcal{F}\mathcal{F}^\star = \mu S$ *and* $[\mathcal{F}, S] = 0$

Example: $q^{n+1} = 1$ $\quad \mathcal{C} = \mathbb{Z}/(n+1)$-graded spaces, $\qquad B = k[x]/(x^{n+1})$

$$\Psi(x^m \otimes x^p) = q^{mp} x^p \otimes x^m \qquad \Delta x = x \otimes 1 + 1 \otimes x \qquad \Longrightarrow \quad \int x^m = \delta_{m,n}$$

$$B^\star = k[y]/(y^{n+1}) \qquad \mathrm{ev}(y^m \otimes x^p) = \delta_{m,p}[m;q]! \qquad \exp = \textstyle\sum_{m=0}^n x^m \otimes y^m/[m;q]!$$

$$\mathcal{F}(x^m) = \int x^m \exp(x \otimes y) = \frac{y^{n-m}}{[n-m;q]!}, \quad \mathcal{F}^\star(y^m) = \frac{q^{(n-m)^2} x^{n-m}}{[n-m;q]!}, \quad \mu = [n;q]!^{-1}$$

$$\mathcal{F}\mathcal{F}^\star = q^{2D+1}\mu S \qquad S\mathcal{F} = \mathcal{F}Sq^{2D+1} \qquad D = \text{monomial deg}$$

Apply to quantum geometry of a Hopf algebra H

S.M. Alg. Repn. Th. 2017

$\Omega(H)$ is a super Hopf algebra (Brzezinski),

$\Omega\cong H{\ltimes}\Lambda$ $\quad$ Λ a super braided Hopf algebra in $\mathcal{C}=\overset{\times}{\mathcal{M}}{}^{H}_{H}=\mathcal{Y}D^{H}_{H}$ with primitive generators

if Vol, g central and binvariant => $\quad \#=(\mathrm{id}\otimes g\circ\mathcal{F}):\Omega^{n}\to\Omega^{top-n}$

Lemma $\quad \mu=\langle\mathrm{Vol},\mathrm{Vol}\rangle^{-1}\in k^{\times}, \quad \exists\sharp^{-1}, \quad \sharp S=(-1)^{top}S\sharp$

g quantum symmetric =>

$$S=(-1)^{D},\quad \sharp^{*}=\sharp,\sharp^{2}=\mu,\quad \text{on}\quad D=0,1,top-1,top$$

$$\exp=1\otimes 1+g+\cdots+g^{(n-1)}+\mu\mathrm{Vol}\otimes\mathrm{Vol}$$

codifferential and Hodge Laplacian

$$\delta:=(S\sharp)^{-1}\mathrm{d}(S\sharp),\quad \Box:=\mathrm{d}\delta+\delta\mathrm{d}$$

$$\mathrm{d}f=(\partial^{a}f)e_{a},\quad \alpha=\alpha^{a}e_{a},\quad g=g^{ab}e_{a}\otimes e_{b}$$

=> $\quad \delta\alpha=\alpha^{a}\delta e_{a}+g_{ab}\partial^{a}\alpha^{b},\quad \Box f=(\partial^{a}f)\delta e_{a}+g_{ab}\partial^{a}\partial^{b}f$

=> canonical Hodge operator:

$$\exp = \sum_{m=0}^{top} \sum_{I,J} e_{i_1} \cdots e_{i_m} ({}_m B)^{-1}_{IJ} \otimes e_{j_1} \cdots e_{j_m}$$

$${}_m B_{IJ} = \langle e_{i_1} \cdots e_{i_m}, e_{j_1} \cdots e_{j_m} \rangle = \mathrm{ev}(e_{i_1} \otimes e_{i_2} \cdots \otimes e_{i_m}, [m, -\tilde{\Psi}]!(e_{j_1} \otimes e_{j_2} \cdots e_{j_m})$$

$$= g_{i_1 p_1} \cdots g_{i_m p_m} [m, -\tilde{\Psi}]!^{p_m \cdots p_2 p_1}_{j_1 j_2 \cdots j_m}$$

=>

$$\sharp e_a = -q^4 e_a e_b e_c, \quad \sharp e_b = -q^4 e_a e_b e_d, \quad \sharp e_c = q^6 e_a e_c e_d, \quad \sharp e_d = q^4 e_b e_c e_d + \lambda q^4 e_a e_b e_c$$

$$\sharp(e_a e_b) = -q^2 e_a e_b, \quad \sharp(e_a e_c) = q^4 e_a e_c, \quad \sharp(e_a e_d) = q^2 e_b e_c + \lambda q^4 e_a e_d$$

$$\sharp(e_b e_c) = q^4 e_a e_d, \quad \sharp(e_b e_d) = q^4 e_b e_d + (1 - q^4) e_a e_b, \quad \sharp(e_c e_d) = -q^2 e_c e_d$$

$$\sharp(e_a e_b e_c) = -q^2 e_a, \quad \sharp(e_a e_b e_d) = -q^2 e_b \quad \sharp(e_a e_c e_d) = e_c \quad \sharp(e_b e_c e_d) = q^2 e_d + \lambda q^2 e_a$$

$$\lambda = 1 - q^{-2}$$

=> on $\Omega^D(\mathbb{C}_q[SL_2])$

$$\sharp^2 = q^6, \quad (D \neq 2); \quad (\sharp - q^4)(\sharp + q^2) = 0, \quad (D = 2).$$

up to normalisation obeys the q-Hecke relation and $\square|_A = \Delta_q$

3. Poisson-Riemannian Geometry

w/Beggs J. Geom. Phys. 2017

$A_0 = C^\infty(M)$ quantisation at order λ means a Poisson bracket

$$a.b - b.a = \lambda\{a,b\} + O(\lambda^2)$$

$\{\ ,\ \} \leftrightarrow \omega^{ij}$ Poisson tensor

Similarly, quantization of $\Omega^1(M)$ at order λ implies new physical field:

$$a.\mathrm{d}b - (\mathrm{d}b).a = \lambda\nabla_{\hat a}\mathrm{d}b + O(\lambda^2)$$

$$\hat a = \{a,\ \}$$

$\Longrightarrow$ ∇ a Poisson pre-connection along Hamiltonian vec. fields

$$\nabla_{\hat a}(b\mathrm{d}c) = \{a,b\}\mathrm{d}c + b\nabla_{\hat a}\mathrm{d}c \qquad \mathrm{d}\{a,b\} = \nabla_{\hat a}\mathrm{d}b - \nabla_{\hat b}\mathrm{d}a$$

- At order λ^2 the bimodule associativity is $(\nabla_{\hat a}\nabla_{\hat b} - \nabla_{\hat b}\nabla_{\hat a} - \nabla_{\widehat{\{a,b\}}})\mathrm{d}c = 0$

$$(\text{just consider } [a,[b,\mathrm{d}c]] + [b,[\mathrm{d}c,a]] + [\mathrm{d}c,[a,b]] = 0)$$

non-flat connection => nonassociativity at $O(\lambda^2)$ not at order λ

- Equiv to Lie Rinehart connection [Huebschmann '90]
  also called `contravariant connection' [Hawkins]

$$\nabla_{\hat a} = \nabla_{\mathrm{d}a}$$

Suppose ∇ an actual connection restricting to $\nabla_{\hat{a}}$ $\widehat{\nabla}$ l.c. of classical g

(1) $\nabla g = 0$ <=> $\widehat{\nabla} = \nabla + S$ $S^a_{bc} = \frac{1}{2} g^{ad}(T_{dbc} - T_{bcd} - T_{cbd})$

=> \`quant metric' $g_1 := q^{-1}(g - \frac{\lambda}{4} g_{ij}\omega^{is}(T^j_{nm;s} - R^j{}_{nms} + R^j{}_{mns})\mathrm{d}x^m \otimes_0 \mathrm{d}x^n)$

(2) Poisson compat <=> $(\widehat{\nabla}_k \omega)^{ij} + \omega^{ir} S^j_{rk} - \omega^{jr} S^i_{rk} = 0$

=> \`quant wedge product'

(3) $\widehat{\nabla}\mathcal{R} + \omega^{ij} g_{rs} S^s_{jn}(R^r{}_{mki} + S^r_{km;i})\,\mathrm{d}x^k \otimes \mathrm{d}x^m \wedge \mathrm{d}x^n = 0$

$\mathcal{R} = g_{ij}\omega^{is}(T^j_{nm;s} - 2R^j{}_{nms})\mathrm{d}x^m \wedge \mathrm{d}x^n$

=> \`quantum levi-civita conn' => $\Delta_1 f = \Delta f + \frac{\lambda}{2}\omega^{\alpha\beta}(\mathrm{Ric}^\gamma{}_\alpha - S^\gamma{}_{;\alpha})(\widehat{\nabla}_\beta \mathrm{d}f)_\gamma$

Thm. $\exists$ monoidal functor to O(λ)
Q: Bundles w. Connection $\longrightarrow$ A-Bimodules w. bimodule Connection

$Q(E) = E$ **but with deformed product** $\forall a \in A,\ e \in E$

$a \bullet e = a\,e + \frac{\lambda}{2}\,\omega^{ij}\,a_{,i}\,(\nabla_{E j} e) + O(\lambda^2)$ $e \bullet a = a\,e - \frac{\lambda}{2}\,\omega^{ij}\,a_{,i}\,(\nabla_{E j} e) + O(\lambda^2)$ ... etc

Example Sphere

$$\sum (z^i)^2 = 1 \qquad \nabla = \widehat{\nabla} \text{ (the Levi-Civita connection) so } S = 0$$

$$\omega = \mathrm{Vol}^{-1}$$

=>

$$[z^i, z^j]_\bullet = \lambda \epsilon^{ij}{}_k z^k, \quad [z^i, \mathrm{d}z^j]_\bullet = \lambda z^j \epsilon^i{}_{mn} z^m \mathrm{d}z^n$$

associative algebra U(su2), non associative diff calculus due to curvature

$$\begin{aligned} g_1 =& g_{\mu\nu}\mathrm{d}z^\mu \otimes_1 \mathrm{d}z^\nu - \frac{\lambda}{2(z^3)^2}\mathrm{d}z^3 \otimes_1 \epsilon_{3ij} z^i \mathrm{d}z^j + \lambda\widetilde{\mathrm{Vol}} \\ =& g_{\mu\nu}\mathrm{d}z^\mu \otimes_1 \mathrm{d}z^\nu + \frac{\lambda}{2(z^3)^2}\epsilon_{3ij}\left(z^3\mathrm{d}z^i \otimes_1 \mathrm{d}z^j - z^i \mathrm{d}z^3 \otimes_1 \mathrm{d}z^j\right) \end{aligned}$$

$$\begin{aligned} \nabla_1 \mathrm{d}z^\mu &= -z^\mu \bullet g_1 = -\widehat{\Gamma}^\mu{}_{\alpha\beta}\mathrm{d}z^\alpha \otimes_1 \mathrm{d}z^\beta - \lambda z^\mu \widetilde{\mathrm{Vol}} + \frac{\lambda}{2}\left(\mathrm{d}z^3 \otimes_1 (\epsilon^{\mu\beta} g_{\beta\gamma} + \frac{z^\mu z^\beta}{(z^3)^2}\epsilon_{\beta\gamma})\mathrm{d}z^\gamma\right) \\ &= -\widehat{\Gamma}^\mu{}_{\alpha\beta}\mathrm{d}z^\alpha \otimes_1 \mathrm{d}z^\beta - \frac{\lambda}{2(z^3)^2}\left(\epsilon_{3ij} z^\mu z^3 \mathrm{d}z^i \otimes_1 \mathrm{d}z^j - \epsilon^\mu{}_{\nu 3}\mathrm{d}z^3 \otimes_1 \mathrm{d}z^\nu\right) \end{aligned}$$

=> $\mathrm{Ricci}_1 = -\frac{1}{2} g_1$ $\qquad \Delta_1 = \Delta$ undeformed at first order

Uniqueness theorem

w/Fritz Class. Qua. Grav 2017

$$g = a^2(r,t)\mathrm{d}t\otimes\mathrm{d}t + b^2(r,t)\mathrm{d}r\otimes\mathrm{d}r + c^2(r,t)(\mathrm{d}\theta\otimes\mathrm{d}\theta + \sin^2(\theta)\mathrm{d}\phi\otimes\mathrm{d}\phi)$$

generic spherically symmetric metric => unique quantisation to $O(\lambda^2)$

$$S_{022} = c\partial_t c,\quad S_{122} = c\partial_r c,\quad S_{033} = c\partial_t c\sin^2(\theta),\quad S_{133} = c\partial_r c\sin^2(\theta)$$

$$S_{120} = S_{123} = S_{223} = S_{320} = S_{130} = S_{132} = S_{230} = S_{233} = 0$$

=> $r,\ t,\ \mathrm{d}r,\ \mathrm{d}t$ central \`unquantized radius and time' and at each *r,t*

$$[z^i, z^j] = \lambda\epsilon^{ij}{}_k z^k,\quad [z^i, \mathrm{d}z^j] = \lambda z^j\epsilon^i{}_{mn}z^m\mathrm{d}z^n \qquad \sum(z^i)^2 = 1$$

\`non associative fuzzy sphere' as above

E.g. Schwarzschild black hole; $\quad \mathrm{Ric}_1 = O(\lambda^2)$

If $\lambda = \hbar$ this could apply to quantum mechanics …
If $\lambda = \lambda_P$ this is a new paradigm of semi-classical quantum gravity